Multi-phase quantum estimation in the regime of limited data

Jesús Rubio

Department of Physics & Astronomy

University of Sussex

Quantum Talks

16th October 2018

Outline

Quantum networks for multi-parameter estimation

Simultaneous and independent strategies: a global approach

Local strategies

A single shot with moderate prior knowledge

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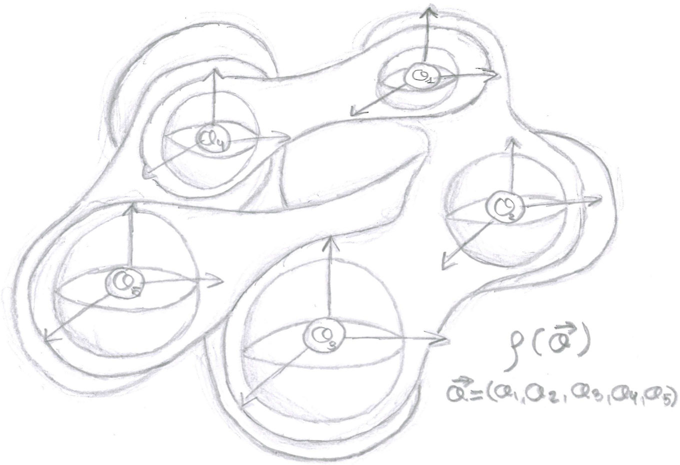
Quantum networks for multi-parameter estimation



Motivation

Fundamental physics

- A better understanding of quantum uncertainty and our capability to extract information (non-commuting estimators, entanglement)
- Testing the limits of our best theories through their use in the design of precise experiments
- Uncovering new physics

[1] M. Szczykulska, Advances in Physics: X, 1:4, 621-639 (2016).

[2], J. A. Dunningham, Contemp. Phys., **47**, 257-267 (2006).

Practical interest: design of *quantum sensing networks*

- Optical interferometry
- Networks of clocks in space
- Quantum imaging
- Estimation of magnetic fields
- Tracking devices based on quantum radar and lidar technologies

[3] T. J. Proctor, Phys. Rev. Lett., **120**, 080501 (2018).
[4] P. A. Knott et al., Phys. Rev. A, **94**, 062312 (2016).
[5] Q. Zhuang et al., Phys. Rev. A, **96**, 040304 (2017).

Step 1: probe preparation

$$\text{Experimental arrangement} \longrightarrow \rho_0$$

Step 2: encoding of a collection of d unknown parameters

$$\rho_0 \longrightarrow \rho(\boldsymbol{\theta}) = U(\boldsymbol{\theta})\rho_0 U^\dagger(\boldsymbol{\theta}),$$

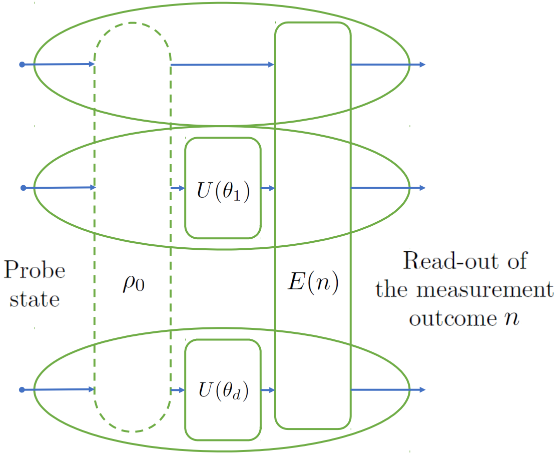
$$\text{where } U(\boldsymbol{\theta}) = U(\theta_1) \otimes \cdots \otimes U(\theta_d)$$

Step 3: measurement scheme and data read-out

$$E(n) \longrightarrow \text{outcome } n,$$

$$\text{with probability } p(n|\boldsymbol{\theta}) = \mathrm{Tr}\left[E(n)\rho(\boldsymbol{\theta})\right]$$

Our case: multi-phase estimation protocol



Encoding of the vector parameter $\boldsymbol{\theta}$

Parameter information summary

$$\text{prior } p(\boldsymbol{\theta}),\ \text{likelihood } p(n|\boldsymbol{\theta}) \longrightarrow p(\boldsymbol{\theta}, n) = p(\boldsymbol{\theta})p(n|\theta)$$

Parameter estimation

$$p(\boldsymbol{\theta}, n) \longrightarrow \begin{cases} \text{estimate} : \boldsymbol{g}(n) = [g_1(n), \ldots, g_d(n)] \\ \text{measure of uncertainty} : \sqrt{\bar{\epsilon}} \end{cases}$$

[6] C.W. Helstrom, Academic Press, New York (1976).

[7] E. T. Jaynes, Cambridge University Press (2003).

Step 4: data processing

Suppose that we record the outcomes $\boldsymbol{n} = (n_1, \ldots, n_\mu)$ after repeating the experiment μ times. Then Bayes' theorem states that

$$p(\boldsymbol{\theta}|n_1, \ldots, n_\mu) = \frac{p(\boldsymbol{\theta})p(n_1|\boldsymbol{\theta}) \cdots p(n_\mu|\boldsymbol{\theta})}{p(n_1, \ldots, n_\mu)}$$

The uncertainty associated to the vector estimator $\boldsymbol{g}(n) = [g_1(n), \ldots, g_d(n)]$ is

$$\int d\boldsymbol{\theta} p(\boldsymbol{\theta}|\boldsymbol{n}) \left\{ \frac{1}{d} \sum_{i=1}^{d} \left[g_i(\boldsymbol{n}) - \theta_i\right]^2 \right\}$$

Optimal strategies for phase-estimation

Strategy: estimators $\boldsymbol{g}(\boldsymbol{n})$ + measurement scheme $E(n_1), \ldots, E(n_\mu)$, given

a) a flat prior for $\theta_i \in [-W_0/2, W_0/2]$, $i = 1, \ldots, d$,

b) a initial state ρ_0,

c) a unitary encoding $U(\boldsymbol{\theta}) = \exp\left(-i \sum_{i=1}^{d} H_i \theta_i\right)$ with generators H_i (Hamiltonian operators),

d) and a fixed averaged energy $\langle H_0 \rangle + \sum_{i=1}^{d} \langle H_i \rangle = \bar{n}$

When we design experiments we do not know the specific outcomes $\boldsymbol{n} = (n_1, \ldots, n_\mu)$ in advance. Therefore, our measure of uncertainty needs to be

$$\bar{\epsilon} = \int d\boldsymbol{n} p(\boldsymbol{n}) \int d\boldsymbol{\theta} p(\boldsymbol{\theta}|\boldsymbol{n}) \left\{ \frac{1}{d} \sum_{i=1}^{d} \left[g_i(\boldsymbol{n}) - \theta_i \right]^2 \right\}$$

For optical phases: valid in the regime of moderate prior knowledge ($W_0 \lesssim 2$).

[8] Jesús Rubio and Jacob Dunningham, "Quantum metrology in the regime of limited data", *in preparation*.

[9] Jesús Rubio et al., J. Phys. Comm., 2(1):015027 (2018).

Optimisation based on the *quantum Cramér-Rao bound*:

$$\bar{\epsilon} \gtrsim \frac{1}{\mu d} \sum_{i=1}^{d} \left(F^{-1}\right)_{ii}$$

where

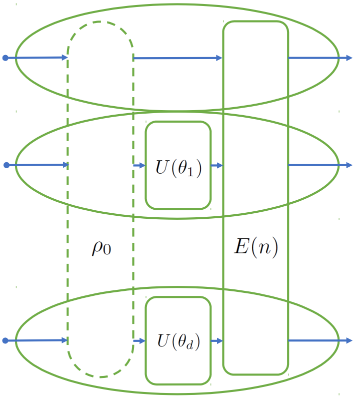
$$F_{ij} = 4\left(\langle H_i H_j \rangle - \langle H_i \rangle \langle H_j \rangle\right)$$

are the elements of the *quantum Fisher information matrix* for pure states and commuting generators.

Only valid in general when the number of repetitions μ is very large!

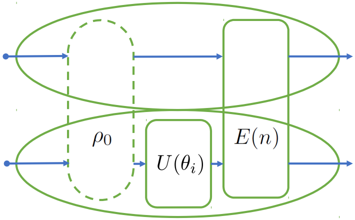
[10] T. Proctor et al., arXiv: 1702.04271 (2017).
[11] S. Ragy et al., Phys. Rev. A **94**, 052108 (2016).

Simultaneous and independent strategies: a global approach



Simultaneous strategy

vs.



d independent schemes

1. d independent NOON states with $\bar{n}/d$ photons each

$$|\psi_0\rangle = \frac{1}{\sqrt{2^d}} \left(|\bar{n}/d\ 0\rangle + |0\ \bar{n}/d\rangle\right)^{\otimes d} \implies \bar{\epsilon} \gtrsim \frac{d^2}{\mu \bar{n}^2}$$

2. Generalised NOON state (optimal version)

$$|\psi_0\rangle = \frac{1}{(\sqrt{d}+d)^{1/2}} (d^{1/4}\,|\bar{n}\ 0 \cdots 0\rangle + |0\ \bar{n}\ 0 \cdots 0\rangle + \cdots + |0 \cdots 0\ \bar{n}\rangle)$$

$$\Downarrow$$

$$\bar{\epsilon} \gtrsim \frac{(1+\sqrt{d})^2}{4\mu\bar{n}} \quad \xrightarrow[d \gg 1]{} \quad \frac{d}{4\mu\bar{n}^2}$$

[12] P.C.Humphreys et al., Phys. Rev. Lett., 111:070403 (2013).

A natural question arises:

Is the entanglement between sensors the cause of this enhancement?

Local strategies

Suppose that we consider a **simultaneous but local** strategy based on the family of probes

$$|\psi_0\rangle = \left[\sqrt{1-\frac{\bar{n}}{N(d+1)}}\,|0\rangle + \sqrt{\frac{\bar{n}}{N(d+1)}}\,|N\rangle\right]^{\otimes d+1}$$

where N is a free parameter that can be varied while $\sum_{i=0}^{d}\langle H_i\rangle = \bar{n}$ remains constant. Then

$$\bar{\epsilon} \gtrsim \frac{(1+d)^2}{4\mu\bar{n}[(1+d)N-\bar{n}]}$$

[4] P. A. Knott et al., Phys. Rev. A, **94**, 062312 (2016).

If $N = \bar{n}$, then

$$\bar{\epsilon} \gtrsim \frac{(1+d)^2}{4\mu\bar{n}^2 d} \quad \xrightarrow[d \gg 1]{} \quad \frac{d}{4\mu\bar{n}^2}$$

which recovers the same scaling that the generalised NOON state. *The enhancement depends on the intra-mode correlations within each mode.*

However, we can also observe that

$$\bar{\epsilon} \gtrsim \frac{(1+d)^2}{4\mu\bar{n}[(1+d)N - \bar{n}]} \quad \xrightarrow[N \to \infty]{} \quad 0$$

This implies that the local strategy can beat any other imaginable scheme (entangled or not).

Is this really the case?

[4] P. A. Knott et al., Phys. Rev. A, **94**, 062312 (2016).

A single shot with moderate prior knowledge

If we optimise the **single-shot uncertainty** instead, we find that

$$\bar{\epsilon} = \int dn\; p(n) \int d\boldsymbol{\theta} p(\boldsymbol{\theta}|n) \left\{ \frac{1}{d} \sum_{i=1}^{d} \left[g_i(n) - \theta_i \right]^2 \right\} \geqslant \frac{1}{d} \sum_{i=1}^{d} \left[\Delta_p \theta_i^2 - \Delta_\rho S_i^2 \right],$$

where S_i is the *optimal quantum estimator* associated to θ_i and $\rho = \int d\boldsymbol{\theta} p(\boldsymbol{\theta}) \rho(\boldsymbol{\theta})$.

[8] Jesús Rubio and Jacob Dunningham, "Quantum metrology in the regime of limited data", *in preparation*.

[13] H. Yuen and M. Lax, IEEE, 19(6):740-750 (1973).

Global estimation revisited

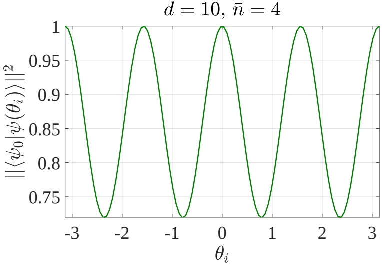
$$|\psi_0\rangle = \frac{1}{(\sqrt{d}+d)^{1/2}}\left(d^{1/4}\,|\bar{n}\ 0\cdots 0\rangle + |0\ \bar{n}\ 0\cdots 0\rangle + \cdots + |0\cdots 0\ \bar{n}\rangle\right)$$

$$\Downarrow$$

$$\bar{\epsilon} \geqslant \frac{1}{\bar{n}^2}\left[\frac{\pi^2}{3} - \frac{4}{(1+\sqrt{d})^2}\right] \underset{d\gg 1}{\longrightarrow} \frac{1}{\bar{n}^2}\left(\frac{\pi^2}{3} - \frac{4}{d}\right)$$

We conclude that generalised NOON states display the same scaling both for $\mu = 1$ and $\mu \gg 1$.

Note that the maximum prior width allowed for unambiguous estimation is $W_0 = 2\pi/\bar{n}$.



Local estimation revisited

$$|\psi_0\rangle = \left[\sqrt{1-\frac{\bar{n}}{N(d+1)}}\,|0\rangle + \sqrt{\frac{\bar{n}}{N(d+1)}}\,|N\rangle\right]^{\otimes d+1}$$

$$\Downarrow$$

$$\bar{\epsilon} \geqslant \frac{1}{\bar{n}}\left[\frac{\pi^2}{3} - f(N,\bar{n},d)\right]$$

The scaling of the global strategy is recovered when $N = \bar{n}$, since in that case

$$\bar{\epsilon} \geqslant \frac{1}{\bar{n}}\left[\frac{\pi^2}{3} - \frac{4d}{(1+d)^2}\right] \underset{d\gg 1}{\longrightarrow} \frac{1}{\bar{n}^2}\left(\frac{\pi^2}{3} - \frac{4}{d}\right)$$

This implies that, a priori, using entanglement is not necessarily a better strategy for a single shot. However...

$$\bar{\epsilon} \geqslant \frac{1}{\bar{n}} \left[\frac{\pi^2}{3} - f(N, \bar{n}, d) \right] \underset{N \to \infty}{\longrightarrow} \frac{1}{d} \sum_{i=1}^{d} \Delta_p \theta_i^2 = \frac{\pi^2}{3\bar{n}}$$

That is, our method removes the non-physical solution. The key to understand why is to observe that the periodicity associated to

$$|\psi_0\rangle = \left[\sqrt{1 - \frac{\bar{n}}{N(d+1)}} \, |0\rangle + \sqrt{\frac{\bar{n}}{N(d+1)}} \, |N\rangle \right]^{\otimes d+1}$$

is $2\pi/N$; consequently, the limit $N \to \infty$ requires to know in advance the solution to the estimation problem.

As a consequence, *there could be other entangled (or non-entangled) schemes that, for a given amount of prior knowledge, provide a better performance that the local strategy proposed in [4]*.

[4] P. A. Knott et al., Phys. Rev. A, **94**, 062312 (2016).

Conclusions

- We have reviewed the fundamentals of multi-parameter quantum estimation theory.

- We have revisited the performance of local and global strategies in the regime of moderate prior knowledge a single shot.

- It has been demonstrated that a rigorous treatment of the prior information removes non-physical solutions and suggests that the privileged status of local strategies requires further study.

Thank you for your attention

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