Quantum sensing networks

A multiparameter approach to the estimation of linear functions

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Key works

J. Phys. A: Math. Theor. **53** 344001
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Phys. Rev. A **101**, 032114
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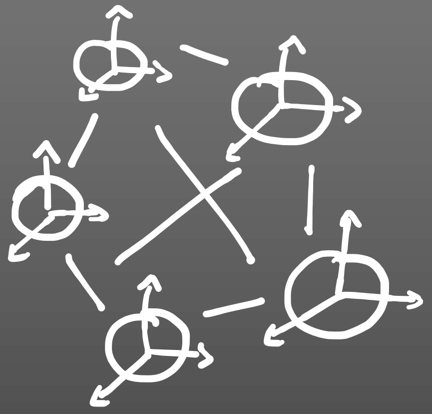

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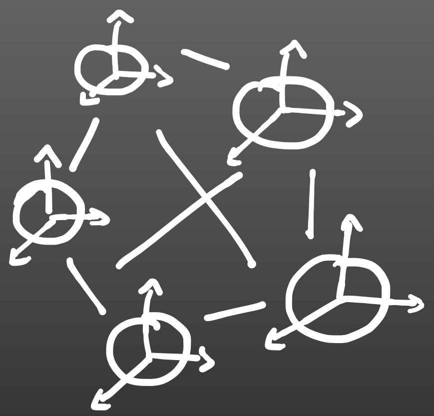
20/05/2021



Collaborators: Paul Knott, Timothy Proctor and Jacob Dunningham

Our plan for today:

1. What information we wish to extract?

   ↳ physical properties, linear functions, and networks: a multiparameter problem

2. How?

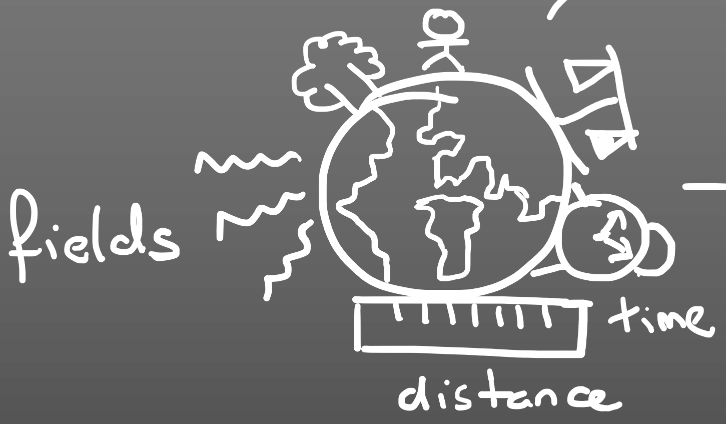
   ↳ qubit sensing networks and the role of inter-sensor correlations

3. The bigger picture: quantum estimation theory à la Bayes

Physical properties, linear functions, and networks

A multiparameter problem

Our description of Nature



direct experience

controlled experiments

$\vec{m} = (m_1, m_2 \ldots)$

physical properties

$\vec{\theta} = (\theta_1, \theta_2, \ldots)$

$\vec{f} = (f_1, f_2, \ldots)$

...

relationships between properties

$f_1(\theta_1, \theta_2)$

$f_{13}(\theta_{11}, \theta_{17}, \theta_{21})$

...

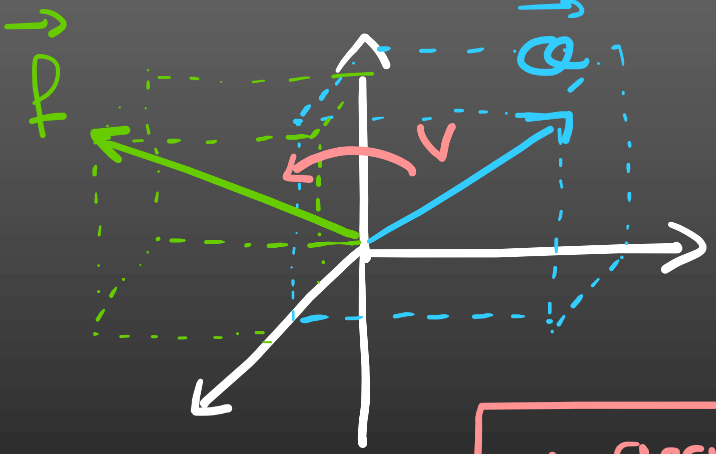
$\vec{f}(\vec{\theta})$

Physical theories

1) Linear functions: exact case

$$\vec{f}(\vec{\theta}) = V^T \vec{\theta} + \vec{a}$$

$$\underset{\ell\times 1}{(\ )} = \underset{\ell\times d}{(\ \ \ )}\,\underset{d\times 1}{(\ )} + \underset{\ell\times 1}{(\ )}$$

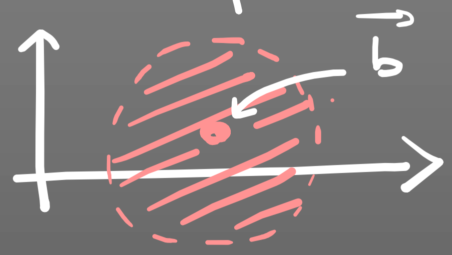


$(\vec{a} = 0)$

$\boxed{V \equiv \text{geometric transformation}}$

2) Linear approximation

* general $\vec{f}(\vec{\theta})$
*



$$\Rightarrow \quad \vec{f}(\vec{\theta}) \simeq \vec{f}(\vec{b}) + \sum_{i=1}^{\ell} \frac{\partial \vec{f}(\vec{b})}{\partial \theta_i}(\theta_i - b_i)$$

$$\equiv V^T \vec{\theta} + \vec{a}$$

QUESTION: If $\vec{\theta} = (\theta_1, \theta_2 \ldots)$ denote spatially separated properties, how can we best estimate (linear) functions $\vec{f} = (f_1, f_2 \ldots)$ of them?

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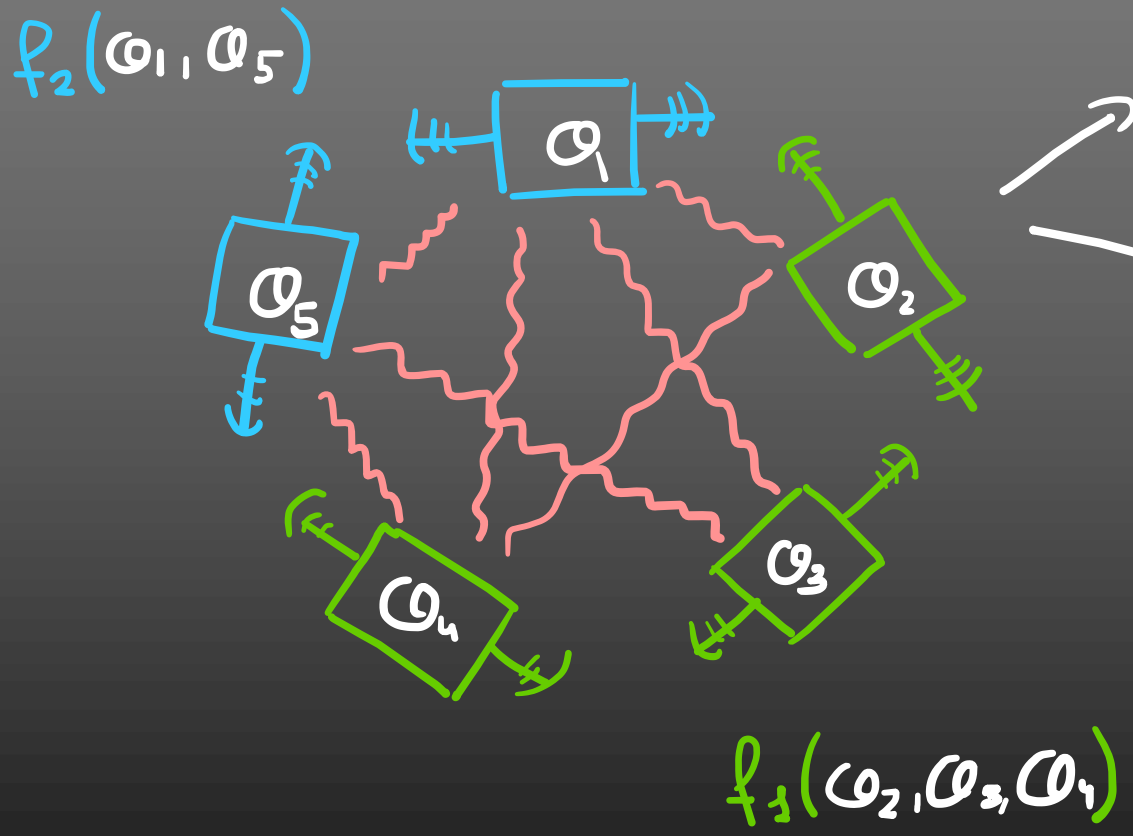


θ_i in each node $=$ $\boxed{\text{local property}}$

f_1, f_2 $=$ $\boxed{\text{global properties}}$

Multiparameter estimation problem

Qubit sensing networks and the role of inter-sensor correlations

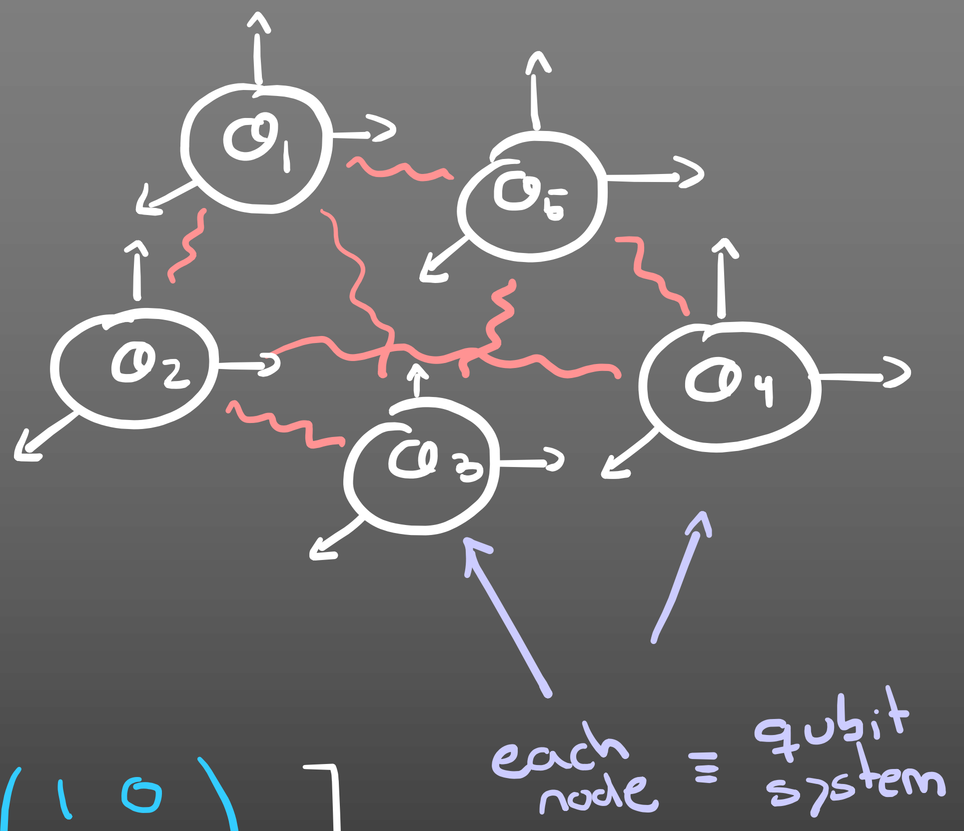
Quantum-enhanced estimation: a qubit sensing network

1) $\rho_0 = |\psi_0\rangle\langle\psi_0|$

$\downarrow$

2) $\rho(\vec{\theta}) = e^{-\vec{K}^T\vec{\theta}}\,\rho_0\,e^{\vec{K}^T\vec{\theta}}$

$$\left[\; e^{-\vec{K}^T\vec{\theta}} \equiv e^{-\sigma_z\theta_1/2} \otimes e^{-\sigma_z\theta_2/2} \otimes \dots;\quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \;\right]$$



* Local vs global strategies

Local $\longrightarrow$ $|\psi_0\rangle = |\psi_0^{(1)}\rangle \otimes |\psi_0^{(2)}\rangle \otimes \cdots$

(same with 1-rank POVMs)

Global $\longrightarrow$ otherwise

* Inter-sensor correlations

$$\mathcal{J}_{ij} \equiv \frac{\langle K_i K_j \rangle - \langle K_i \rangle \langle K_j \rangle}{\Delta K_i \, \Delta K_j} \Longrightarrow$$

For local strategies,

$\mathcal{J}_{ij} = 0, \ \forall i,j$

$$\left(\langle * \rangle = \mathrm{Tr}[\rho_0 \, *]; \ \Delta K_i^2 = \langle K_i^2 \rangle - \langle K_i \rangle^2 \right)$$

* Inter-sensor correlations

$$\mathcal{J}_{ij} \equiv \frac{\langle k_i k_j \rangle - \langle k_i \rangle \langle k_j \rangle}{\Delta k_i \, \Delta k_j}$$

$$\left(\langle * \rangle = \mathrm{Tr}[\rho_0 \, *] \, ; \quad \Delta k_i^2 = \langle k_i^2 \rangle - \langle k_i \rangle^2 \right)$$

→ sensor-symmetric states

[Recent generalisation: arXiv:2104.09540]

$$\langle k_i k_j \rangle - \langle k_i \rangle \langle k_j \rangle \equiv c$$

$$\langle k_i^2 \rangle - \langle k_i \rangle^2 \equiv v \quad \forall i,j$$

$$\Rightarrow \quad \mathcal{J}_{ij} \equiv \mathcal{J} = \frac{c}{v}$$

$$\left\{ \begin{array}{c} \text{state} \longleftrightarrow (v, \mathcal{J}) \\ (\vec{k} \text{ fixed}) \end{array} \right.$$

QUESTION: best network state ρ_0 to estimate $\vec{f}(\vec{\theta})$?

Define the uncertainty quantifier

$$\bar{\epsilon}_{CR} := \frac{1}{\mu} \mathrm{Tr}\left[W V^{T} F_q^{-1} V \right]$$

leads to fundamental limits in a certain sense

Weighting matrix

$W = w_i \delta_{ij}$

Fisher information matrix

$$(F_q)_{ij} = 4\left(\langle k_i k_j \rangle - \langle k_i \rangle \langle k_j \rangle\right)$$

For the qubit sensing network under consideration:

$$\bar{\epsilon}_{cr} = \frac{[1+(d-2)\mathcal{J}]\,\mathrm{Tr}(W V^T V) - \mathcal{J}\,\mathrm{Tr}(W V^T X V)}{4\mu\upsilon(1-\mathcal{J})[1+(d-1)\mathcal{J}]}$$

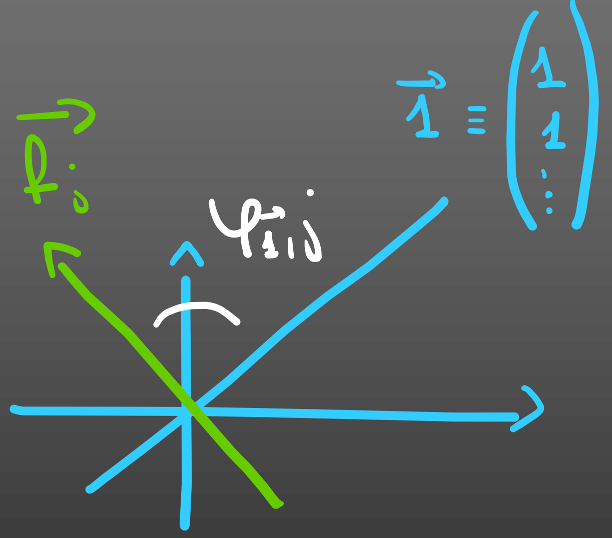
?

where $X = \begin{pmatrix} 0 & 1 & \cdots \\ 1 & 0 & \ddots \\ \vdots & \ddots & \ddots \end{pmatrix}.$

* Geometric reinterpretation:

$$V = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{f}_1 & \vec{f}_2 & \cdots & \vec{f}_\ell \\ \downarrow & \downarrow & & \downarrow \end{pmatrix} \longleftrightarrow f_j(\vec{\omega}) = \sum_{i=1}^{d} V_{ij}\,\omega_i \equiv \vec{f}_j^{\,T}\vec{\omega}$$

$$\Rightarrow \begin{cases} \mathrm{Tr}(W V^T V) = \sum_{j=1}^{\ell} w_j |\vec{f}_j|^2 \\ \mathrm{Tr}(W V^T X V) = \sum_{j=1}^{\ell} w_j |\vec{f}_j|^2 \left[d \cos^2(\varphi_{\vec{1} j}) - 1\right] \end{cases}$$



→ Normalisation term

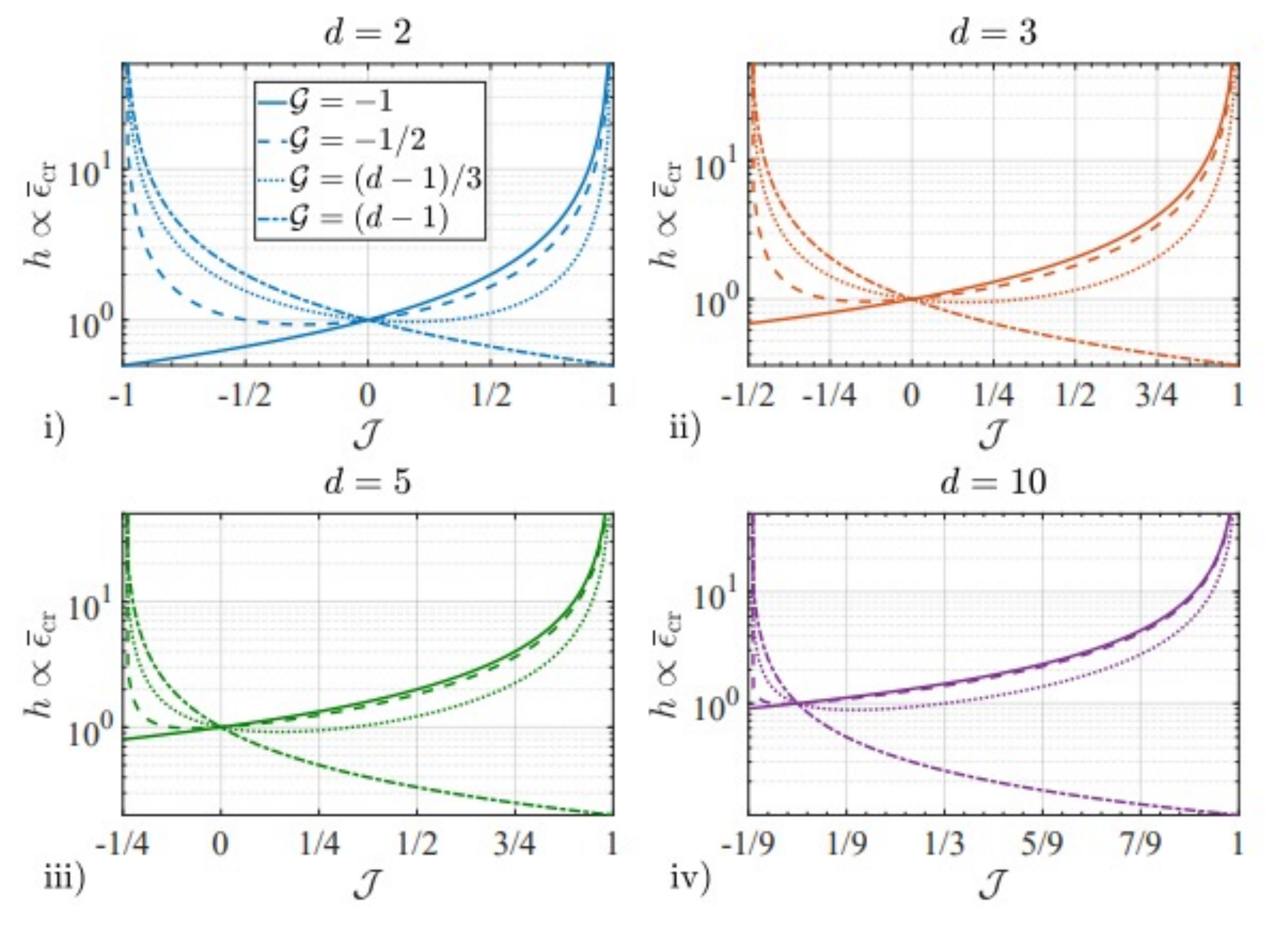
$$\mathcal{N} := \mathrm{Tr}(W V^T V)$$

→ Geometry parameter

$$\mathcal{G} := \frac{1}{\mathcal{N}} \mathrm{Tr}[W V^T X V]$$

→ Final uncertainty

$$\boxed{\bar{\epsilon}_{cr} = \frac{\mathcal{N}}{4M\sigma} \underbrace{\frac{[1+(d-\mathcal{G})\mathcal{J}]}{(1-\mathcal{J})[1+(d-1)\mathcal{J}]}}_{= h(\mathcal{J}, \mathcal{G}, d)}}$$



QUESTION: Given $(N, \mathcal{G})$, optimal $(v, \mathcal{J})$?

$(N, \mathcal{G}) \to$ geometry of linear functions

$(v, \mathcal{J}) \to$ state ρ_0 ($\vec{k}$ fixed)

arXiv:2003.04867 (this talk)

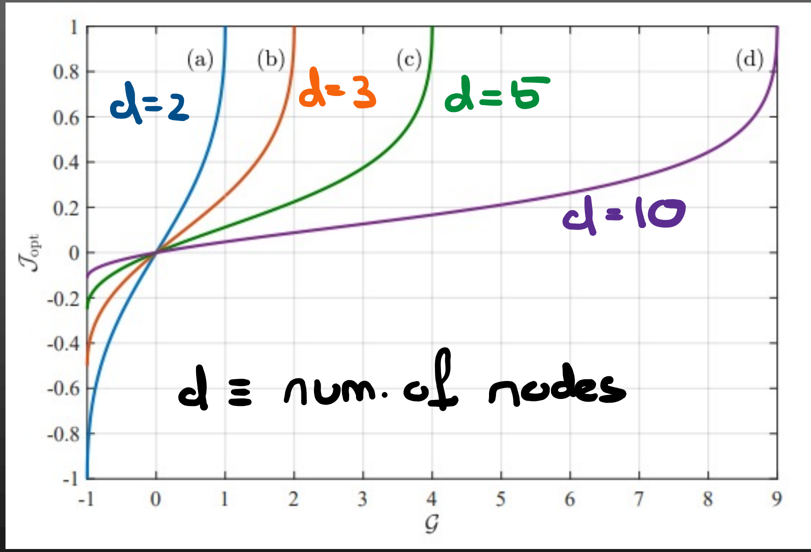
arXiv:2104.09540

$$\bar{\epsilon}_{cr} \geq \frac{N}{\mu} h(\mathcal{J}, \mathcal{G}, d)$$

$v \leq \frac{1}{4}$

$\Rightarrow$ minimisation of h w.r.t. $\mathcal{J}$

$$\boxed{\mathcal{J}_{opt} = \frac{1}{\mathcal{G}+2-d}\left[1-\sqrt{\frac{(\mathcal{G}+1)(d-1-\mathcal{G})}{d-1}}\right]}$$



The bigger picture: Quantum estimation theory à la Bayes

* A closer look to the standard approach

MSE

CCRB

$$\text{①}\quad \int d\vec{m}\, \cancel{p(\vec{m}|\vec{\Theta})}\, \mathrm{Tr}\left[W V^T [\tilde{\vec{\Theta}}(\vec{m}) - \vec{\Theta}][\tilde{\vec{\Theta}}(\vec{m}) - \vec{\Theta}]^T V\right]$$

$\mathrm{Tr}[\Pi(\vec{m})\,\rho(\vec{\Theta})]$

$$\text{②}\quad \cancel{\geq} \frac{1}{\mu} \mathrm{Tr}\left[W V^T [\mathbb{I} + \partial_{\vec{\Theta}} \vec{b}(\vec{\Theta})]\, F^{-1}(\vec{\Theta})\, [\mathbb{I} + \partial_{\vec{\Theta}} \vec{b}(\vec{\Theta})]^T + \underbrace{\vec{b}(\vec{\Theta})\, \vec{b}^T(\vec{\Theta})}_{\text{bias}}\, V\right]$$

$$\text{③}\quad \underset{\cancel{\vec{b}(\vec{\Theta})=0}}{=} \frac{1}{\mu} \mathrm{Tr}\left[W V^T F^{-1}(\vec{\Theta})\, V\right] \longleftarrow \text{unbiased CCRB}$$

$$\text{④}\quad \geq \frac{1}{\mu} \mathrm{Tr}\left[W V^T F_q^{-1}\, V\right] \longleftarrow \text{QCRB}$$

* A more consistent story

(1) $\bar{\epsilon}_{mse} := \int d\vec{m}\, d\vec{\Theta}\; p(\vec{\Theta})\, p(\vec{m}|\vec{\Theta})\, \mathrm{Tr}\left[W V^T [\tilde{\vec{\Theta}}(\vec{m}) - \vec{\Theta}][\tilde{\vec{\Theta}}(\vec{m}) - \vec{\Theta}]^T V\right]$

(2) $\min_{\tilde{\Theta}} \bar{\epsilon}_{mse} \Rightarrow \boxed{\tilde{\vec{\Theta}}_{opt}(\vec{m}) = \int d\vec{\Theta}\, p(\vec{\Theta}|\vec{m})\,\vec{\Theta}}$

(3) $\bar{\epsilon}_{mse}[\tilde{\Theta}_{opt}] = \bar{\epsilon}_{opt}$

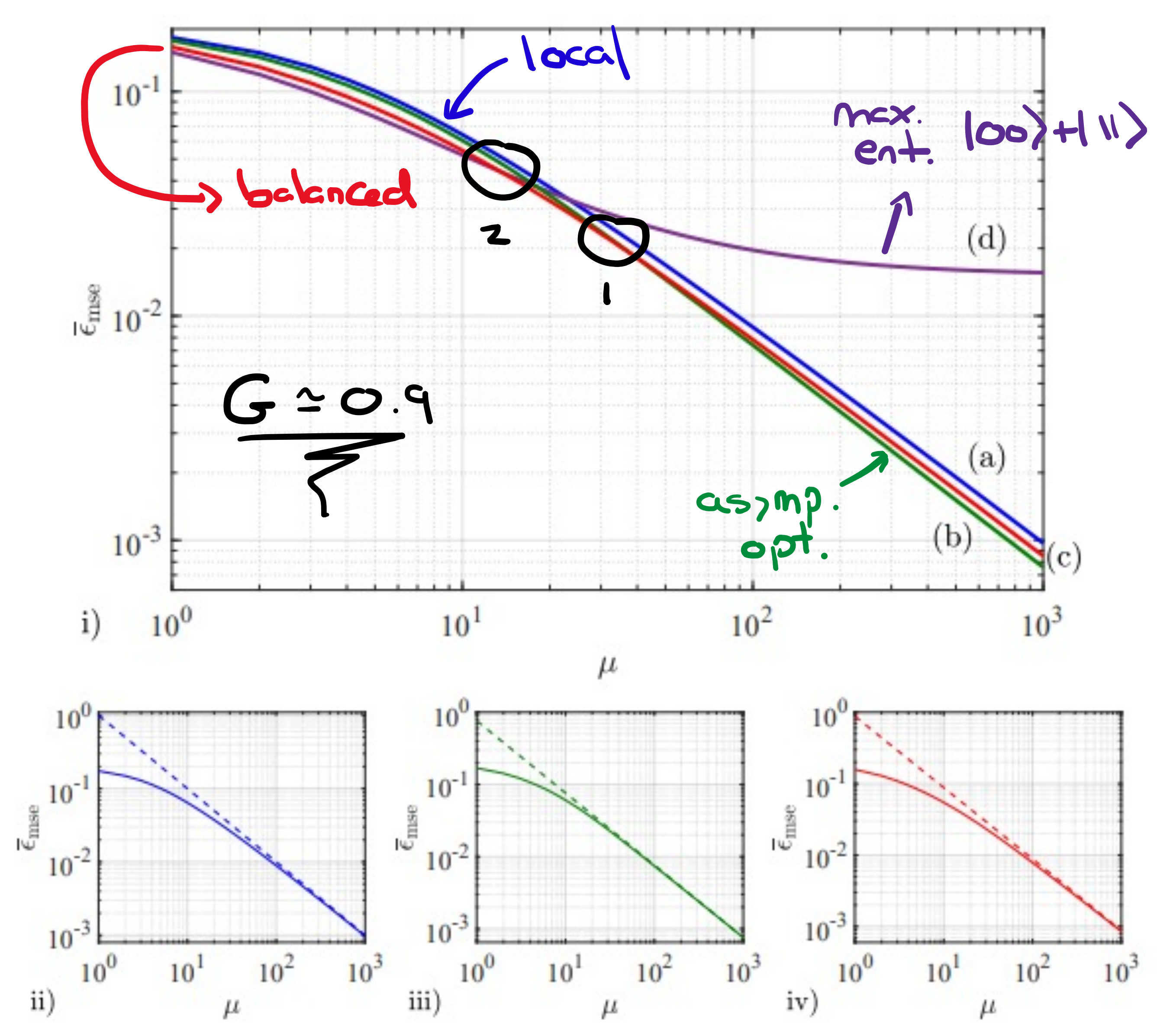
(4) $\bar{\epsilon}_{opt} \underset{\mu \gg 1}{\simeq} \frac{1}{\mu}\int d\vec{\Theta}\, p(\vec{\Theta})\, \mathrm{Tr}\left[W V^T F^{-1}(\vec{\Theta}) V\right]$

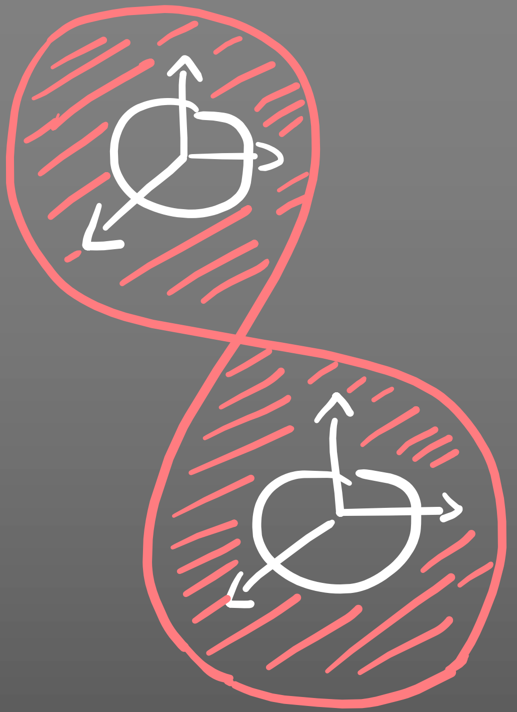
(5) $\boxed{\geq \frac{1}{\mu}\mathrm{Tr}\left[W V^T F_q^{-1} V\right]}$

QCRB

- No $\vec{\Theta}$-dependence ✓
- Explicit prior ✓
- Opt. estimator $\forall \mu$ ✓
- No unbiasedness ✓
- clear assumptions ✓

* Two-qubit network





$f_1(a_1, a_2)$
$f_2(a_1, a_2)$

* Practical optimisation of quantum sensing networks: The future

→ Single-shot Bayesian multiparameter quantum bound: $(V = \mathbb{I})$

$$\bar{\epsilon}_{mse} \geq \sum_{i=1}^{d} w_i \left[\sigma_{0,i}^2 - \Delta S_{\theta,i}^2 \right] \longrightarrow$$

prior uncertainty

opt. quantum estimator

- Ultimate precision when $[S_i, S_j] = 0$ ✓
- It does not account for non-commutativity X

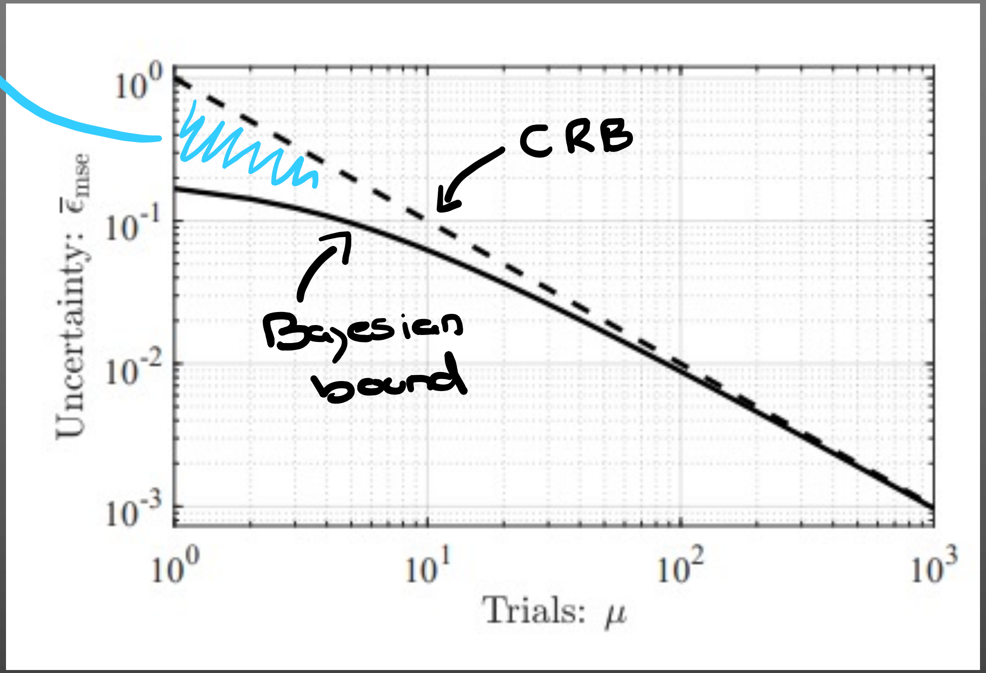
→ Holevo CRB ⟶

- It can handle non-commutativity ✓
- Based on $\vec{\theta}$-dependent uncertainty quantifier X

* Two-qubit network

$\longrightarrow$ θ_1, θ_2 $\longrightarrow$

CRB ⇒ information loss when μ is low

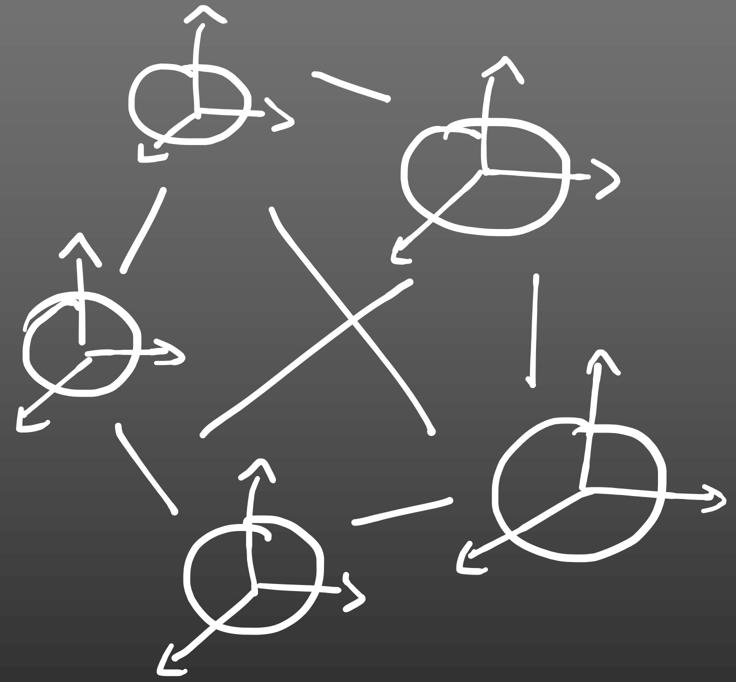


Take-home message:

→ Intimate connection between the correlations of a quantum network and the geometry associated with linear relationships between physical properties.

→ A comprehensive and consistent understanding of optimal protocols for quantum sensing networks will likely require to adopt a general Bayesian approach.

Thank you for your attention!



To learn more:

→ arXiv: 2003.04867

→ arXiv: 1906.04123

→ J. Rubio-Jimenez@exeter.ac.uk