Quantum scale metrology: measuring the lifetime of a mixed state

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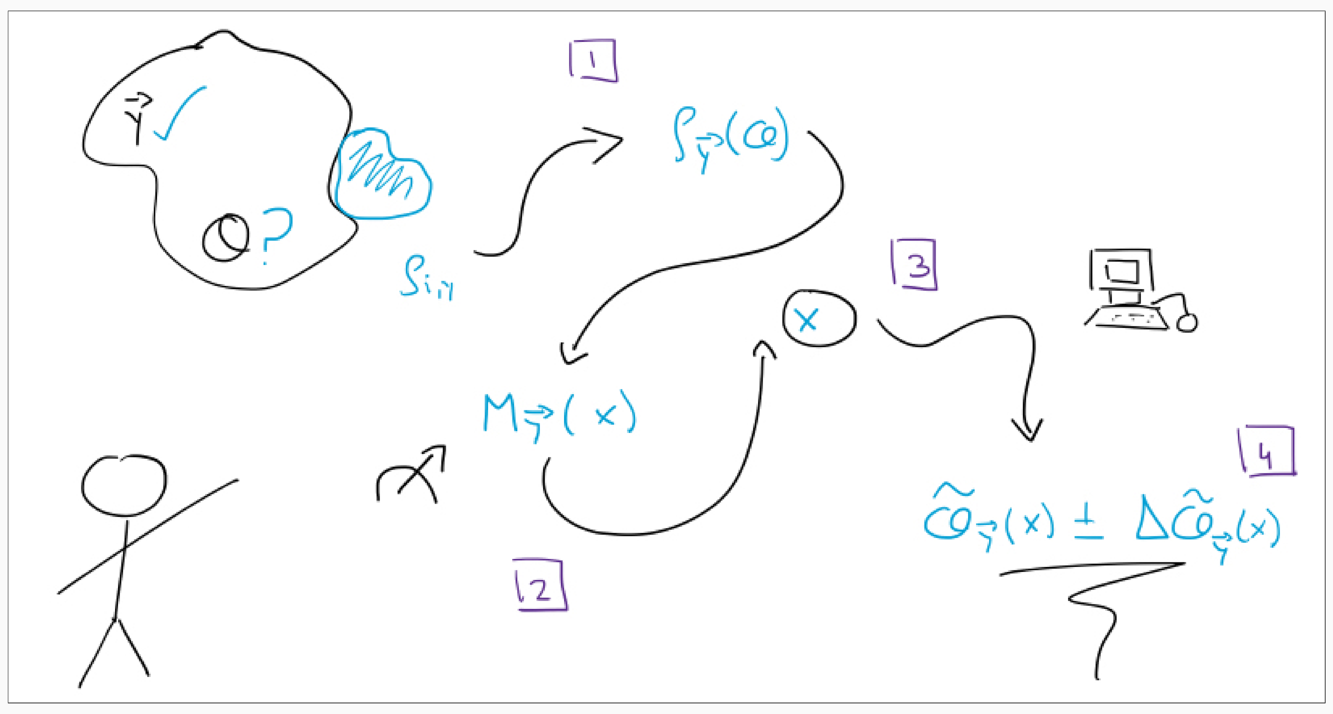


- Progress across modern quantum sciences is intimately connected to the possibility of performing **highly precise measurements**.
- Quantum metrology is being expanded to **new regimes including dissipative dynamics, finite information, incompatible estimators and parameters other than phases**.
- This poster presents the **optimal quantum strategy for the measurement of time scales of dissipative processes**, choosing spontaneous photon emission as a case study.
- This is achieved by means of quantum scale metrology, **a new Bayesian framework based on logarithmic errors that enables the precise estimation of scale parameters**.

Quantum metrology of scale parameters

Experimental description

- Dimensionless measurand x.
- *Known* parameters $\boldsymbol{y} = (y_1, y_2, \dots)$.
- *Unknown* parameter Θ.

What is a scale parameter?

Θ scales y_i if, for fixed Θ, y_i is considered 'large' when $y_i/\Theta \gg 1$ and 'small' when $y_i/\Theta \ll 1$. This is invariant under transformations

$$y_i \mapsto y_i' = \gamma y_i, \quad \Theta \mapsto \Theta' = \gamma\Theta,$$

with positive γ, since $y_i/\Theta = y_i'/\Theta'$.

Metrological protocol

1. Prepare the state $\rho_{\boldsymbol{y}}(\theta)$.
2. Implement the POM $M_{\boldsymbol{y}}(x)$.
3. Record the outcome x, with statistics given by the Born rule
$$\mathrm{Tr}[M_{\boldsymbol{y}}(x)\rho_{\boldsymbol{y}}(\theta)] = h\left(x, \frac{\boldsymbol{y}}{\theta}\right).$$
4. Find the estimator $\tilde{\theta}_{\boldsymbol{y}}(x) \pm \Delta\tilde{\theta}_{\boldsymbol{y}}(x)$.

Optimisation problem

$$\min_{M_{\boldsymbol{y}}(x), \tilde{\theta}_{\boldsymbol{y}}(x)} \mathrm{Tr}\left\{\int dx\, M_{\boldsymbol{y}}(x)\, W_{\boldsymbol{y}}[\tilde{\theta}_{\boldsymbol{y}}(x)]\right\},$$

with prior $p(\theta)$ and

$$W_{\boldsymbol{y}}[\tilde{\theta}_{\boldsymbol{y}}(x)] = \int d\theta\, p(\theta)\, \rho_{\boldsymbol{y}}(\theta) \log^2\left[\frac{\tilde{\theta}_{\boldsymbol{y}}(x)}{\theta}\right].$$

Result 1: Optimal strategy

Let the operator

$$\mathcal{S}_{\boldsymbol{y}} = \int ds\, \mathcal{P}_{\boldsymbol{y}}(s)\, s$$

solve the Lyaponuv equation

$$\mathcal{S}_{\boldsymbol{y}}\varrho_{\boldsymbol{y},0} + \varrho_{\boldsymbol{y},0}\mathcal{S}_{\boldsymbol{y}} = 2\varrho_{\boldsymbol{y},1},$$

where

$$\varrho_{\boldsymbol{y},k} = \int d\theta\, p(\theta)\rho_{\boldsymbol{y}}(\theta)\log^k\left(\frac{\theta}{\theta_u}\right);$$

then, the *optimal estimator* is

$$\tilde{\theta}_{\boldsymbol{y}}(x) \mapsto \tilde{\vartheta}_{\boldsymbol{y}}(s) = \theta_u \exp(s),$$

and the *optimal POM* is

$$M_{\boldsymbol{y}}(x) \mapsto \mathcal{M}_{\boldsymbol{y}}(s) = \mathcal{P}_{\boldsymbol{y}}(s).$$

Result 2: Ultimate precision limits

The hierarchy of inequalities

$$\bar{\epsilon}_{\mathrm{mle}} \geq \bar{\epsilon}_p - \mathcal{K}_{\boldsymbol{y}} \geq \bar{\epsilon}_p - \mathcal{J}_{\boldsymbol{y}}$$

gives *fundamental lower bounds on the precision of scale estimation problems.* Here,

- $\bar{\epsilon}_p$ only depends on the prior,
- using the optimal estimator saturates the first inequality, and
- using the optimal POM saturates the second inequality.

The expressions for $\bar{\epsilon}_p$, $\mathcal{K}_{\boldsymbol{y}}$ and $\mathcal{J}_{\boldsymbol{y}}$ are given in Rubio *et al.*[1,2]

Beyond quantum phase estimation

Parameter	phase	location	scale
Support	$0 \leq \theta < 2\pi$	$-\infty < \theta < \infty$	$0 < \theta < \infty$
Symmetry	$\theta \mapsto \theta' = \theta + 2\gamma\pi,\ \gamma \in \mathbb{Z}$	$\theta \mapsto \theta' = \theta + \gamma,\ \gamma \in \mathbb{R}$	$\theta \mapsto \theta' = \gamma\theta,\ \gamma \in \mathbb{R}_{++}$
Ignorance	$p(\theta) = 1/2\pi$	$p(\theta) \propto 1$	$p(\theta) \propto 1/\theta$
Error $\mathcal{D}(\tilde{\theta}, \theta)$	$4\sin^2[(\tilde{\theta} - \theta)/2]$	$(\tilde{\theta} - \theta)^2$	$\log^2(\tilde{\theta}/\theta)$

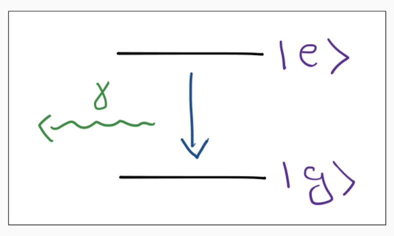
Quantum-enhanced estimation of a time scale

Spontaneous photon emission

Let a two-level atom prepared as

$$|\psi\rangle = \sqrt{1-a}\,|g\rangle + \sqrt{a}\,|e\rangle$$

undergo spontaneous photon emission:



Using the formalism of *open quantum systems*, the statistics of this process may be described as[7]

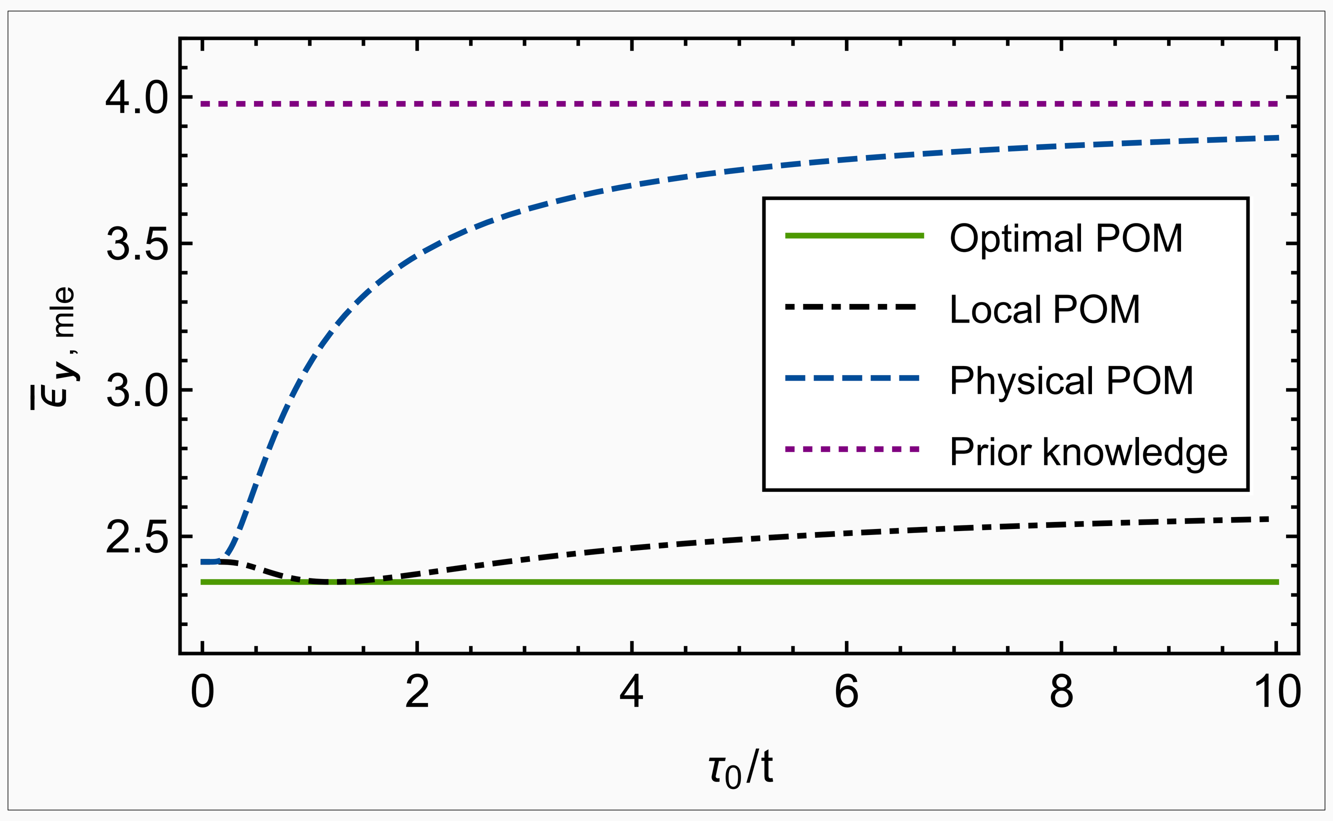
$$\rho_t(\tau) = [1 - a\,\eta_t(\tau)]\,|g\rangle\langle g| + a\,\eta_t(\tau)\,|e\rangle\langle e| + [a(1-a)\,\eta_t(\tau)]^{\frac{1}{2}}\,(|g\rangle\langle e| + |e\rangle\langle g|),$$

with $\eta_t(\tau) := \exp(-t/\tau)$, lifetime τ and elapsed time t.

Estimation problem

Unknown parameter: $\Theta = \tau$; available prior information:

- $\theta/t \in [0.01, 10]$.
- $p(\theta) = 0.145/\theta$.
- $a = 0.9$.



'Yes'/'No' measurement

Whether or not a photon is emitted is captured by the *physical POM*[7]

$$M^Y_{t,\tau_0} = [1 - \eta_t(\tau_0)]\,|e\rangle\langle e| \text{ ('Yes')},$$
$$M^N_{t,\tau_0} = |g\rangle\langle g| + \eta_t(\tau_0)\,|e\rangle\langle e| \text{ ('No')}.$$

Remarks:

- Initial 'hint' τ_o at the true lifetime τ needed.
- Informative, as it reduces the prior uncertainty $\bar{\epsilon}_p$.
- τ easier to estimate when the decay is likely to have already happened, i.e., for $\tau_0/t \ll 1$.
- Yet, generally suboptimal.

SLD measurement

The Fisher information leads to the *local POM*[9]

$$M^i_{t,\tau_0} = |\lambda^i_{t,\tau_0}\rangle\langle\lambda^i_{t,\tau_0}|, \text{ for } i = 1, 2,$$

with $L_t(\tau_0)|\lambda^i_{t,\tau_0}\rangle = \lambda^i_{t,\tau_0}|\lambda^i_{t,\tau_0}\rangle$, $L_t(\tau)\rho_t(\tau) + \rho_t(\tau)L_t(\tau) = 2\partial_\tau\rho_t(\tau)$.

Remarks:

- Initial 'hint' τ_o still needed.
- More informative than 'Yes'/'No' measurements.
- However, suboptimal for $\tau_0/t \gg 1$.

Optimal measurement

The eigendecomposition of $\mathcal{S}_y$ leads to the *optimal POM* $\mathcal{M}^+_y = |\psi_+\rangle\langle\psi_+|$ and $\mathcal{M}^-_y = |\psi_-\rangle\langle\psi_-|$, with

$$|\psi_+\rangle = 0.094\,|g\rangle + 0.996\,|e\rangle,$$
$$|\psi_-\rangle = 0.996\,|g\rangle - 0.094\,|e\rangle.$$

Remarks:

- Globally optimal, τ_0-independent.
- Establishes the fundamental precision limit for the estimation of τ.

Conclusions

- This work demonstrates the **optimal estimation of a time scale using quantum resources.**
- This has been possible thanks to quantum scale metrology, a new framework enabling **the most precise estimation of scale parameters allowed by quantum mechanics.**
- By virtue of having generalised metrology beyond phase estimation, this work sets the path for the **construction of new quantum estimation theories for all kinds of parameters.**

References

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