Global sensing informed by symmetries

A new path to optimality in quantum metrology

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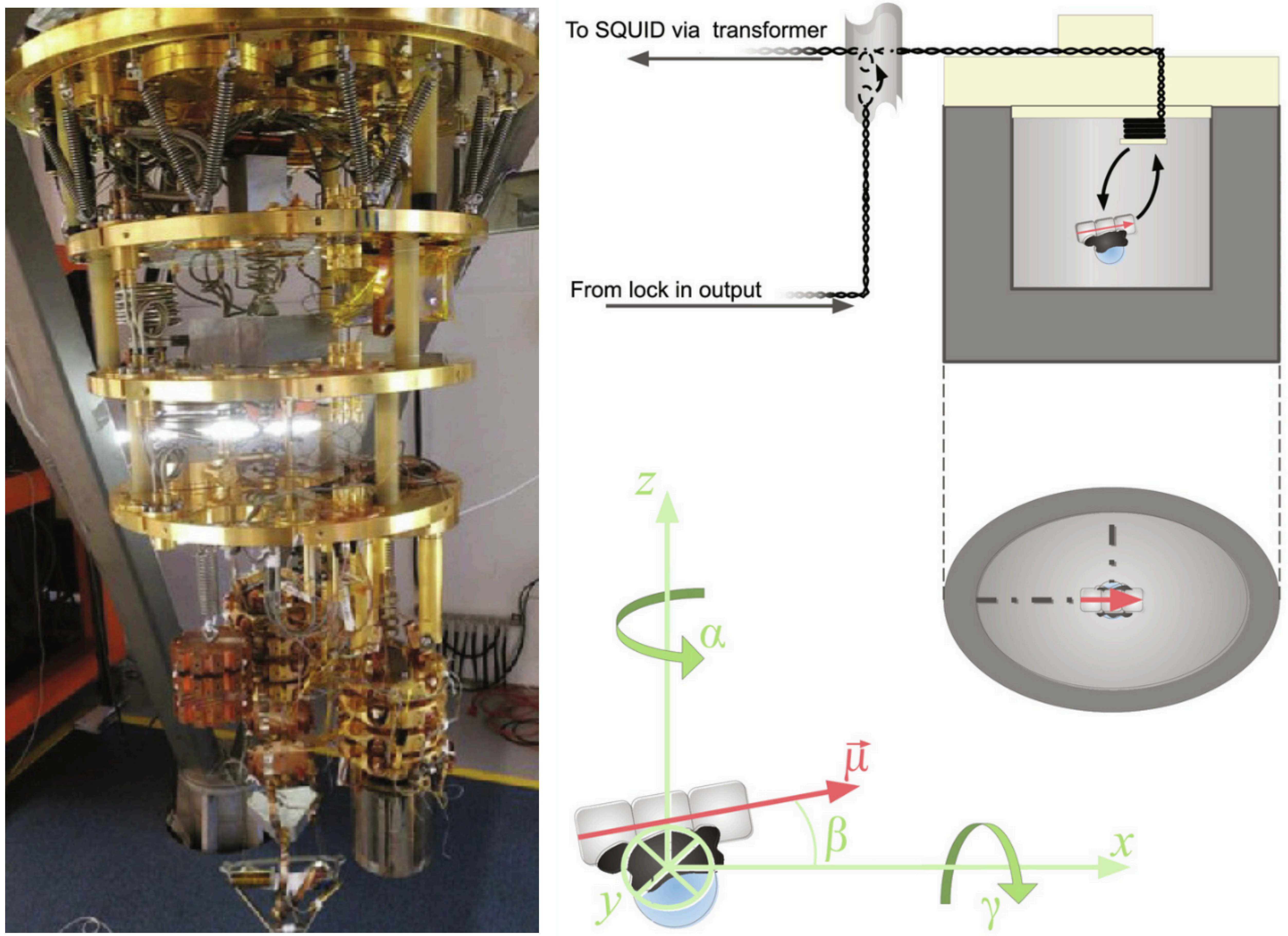
Our plan for today

- Quantum sensing for fundamental physics
- Local and Bayesian tales
- Symmetry leads to optimality
- Experimental validation on cold atom platforms
- The multiparameter challenge: analytical results

Quantum sensing for fundamental physics

Gravity measurements with levitated mechanical systems

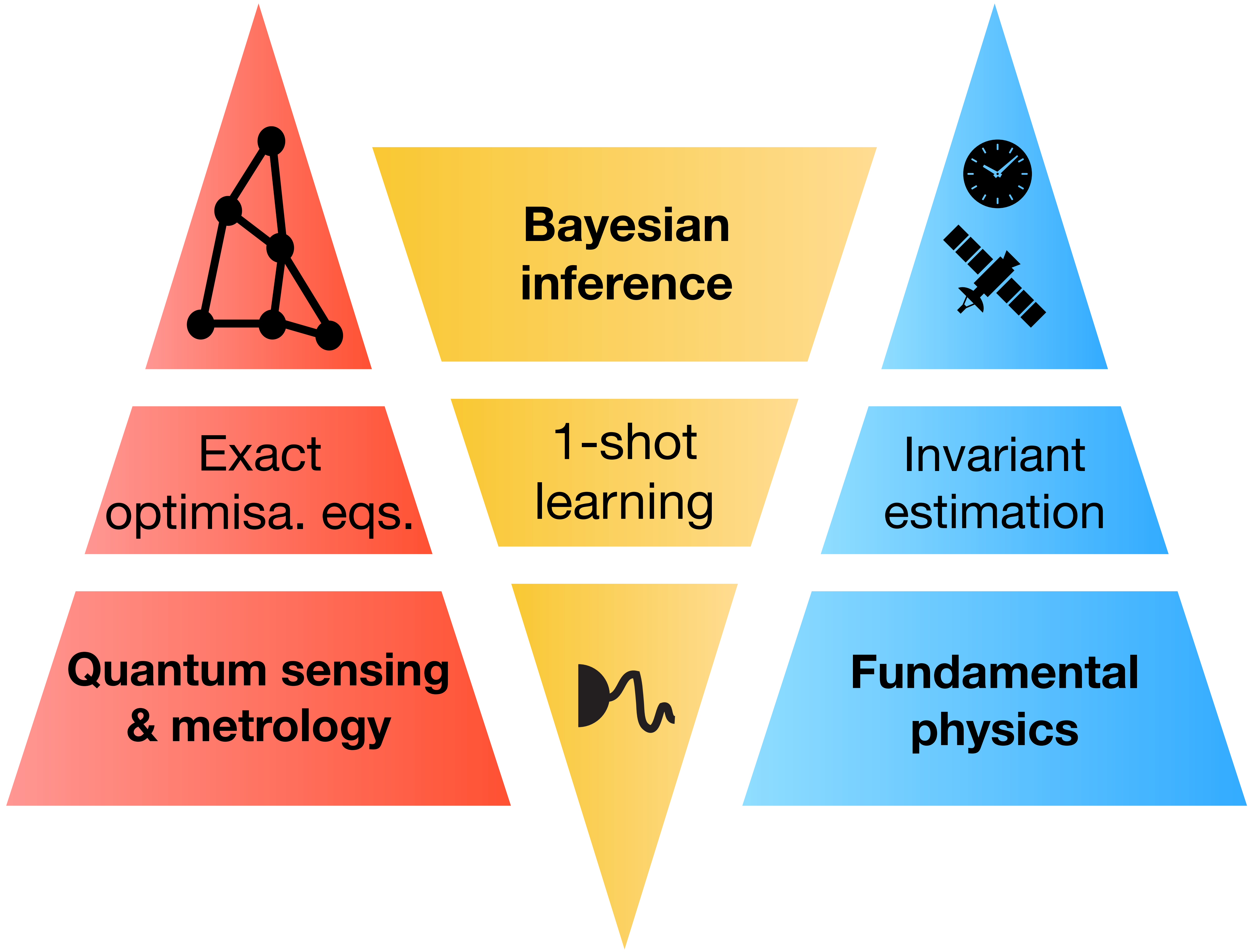


(Fuchs *et al.*, *Sci. Adv.* 10, eadk2949, 2024)

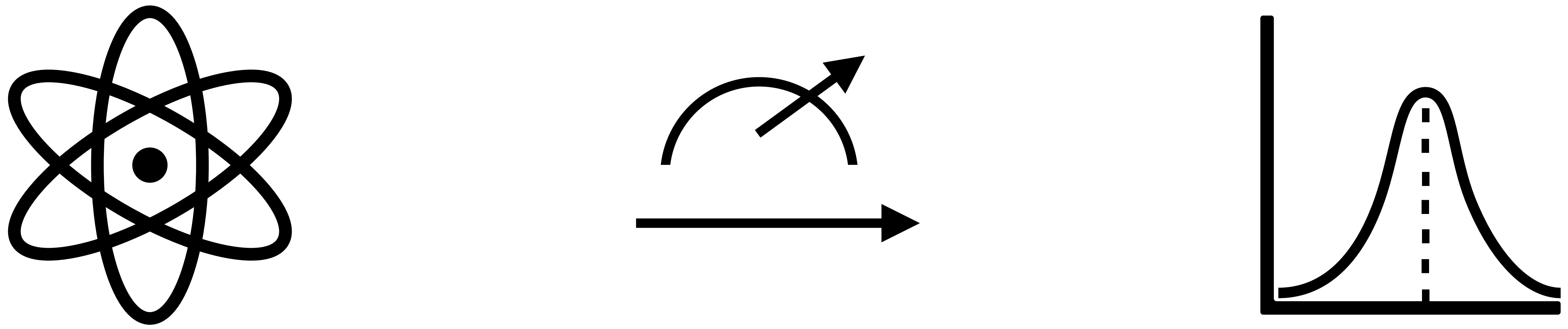
Teledyne e2v are collaborating with STFC RALSpace and University of Birmingham in the development of a cold atom quantum-technology based instrument in preparation for a future space mission to take sensitive measurements of atmospheric drag.

Long-term goal

Designing **universally optimal sensing protocols** that are ***both* quantum *and* Bayesian**, setting a new standard for quantum sensing with **AMO platforms for fundamental physics**.



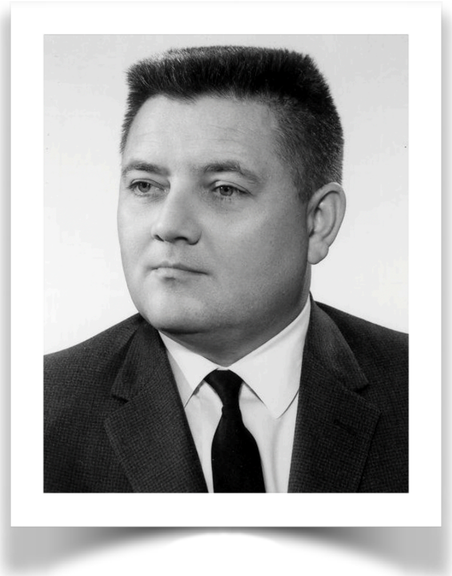
The story so far: local estimation theory



$\rho_{\boldsymbol{y}}(\Theta) \longrightarrow M_{\boldsymbol{y}}(x) \longrightarrow p(\boldsymbol{x}\,|\,\Theta, \boldsymbol{y})$

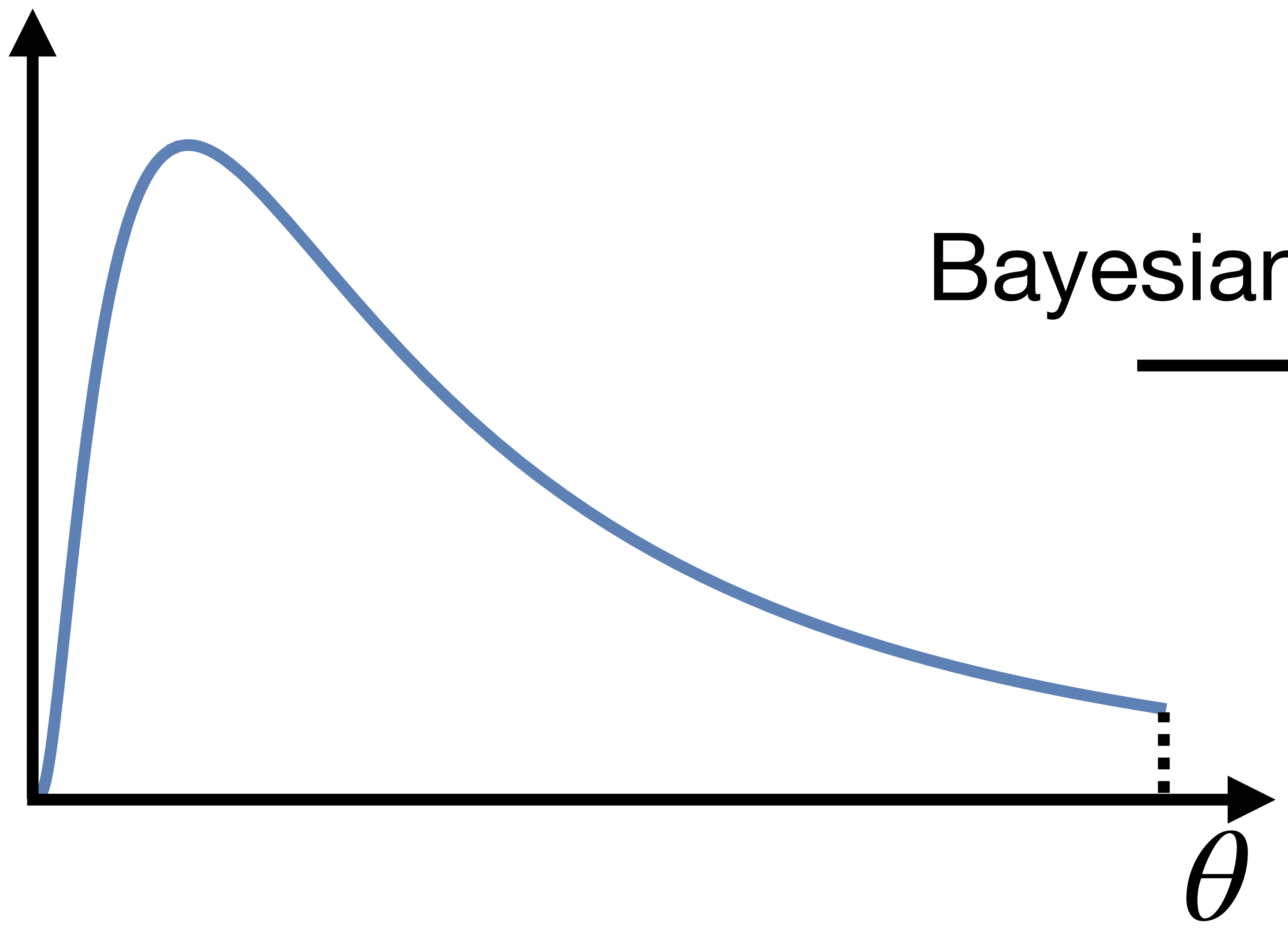
Sensitivity	$F_{\boldsymbol{y}}(\Theta) = Tr[\rho_{\boldsymbol{y}}(\Theta)\, L_{\boldsymbol{y}}(\Theta)]$
Optimal POM (via the SLD)	$L_{\boldsymbol{y}}(\Theta)\,\rho_{\boldsymbol{y}}(\Theta) + \rho_{\boldsymbol{y}}(\Theta)\,L_{\boldsymbol{y}}(\Theta) = 2\,\partial_{\Theta}\rho_{\boldsymbol{y}}(\Theta)$
Estimators	MLE, locally unbiased

Quantum estimation à la Bayes



hypothesis

prior ← $p(\theta)$

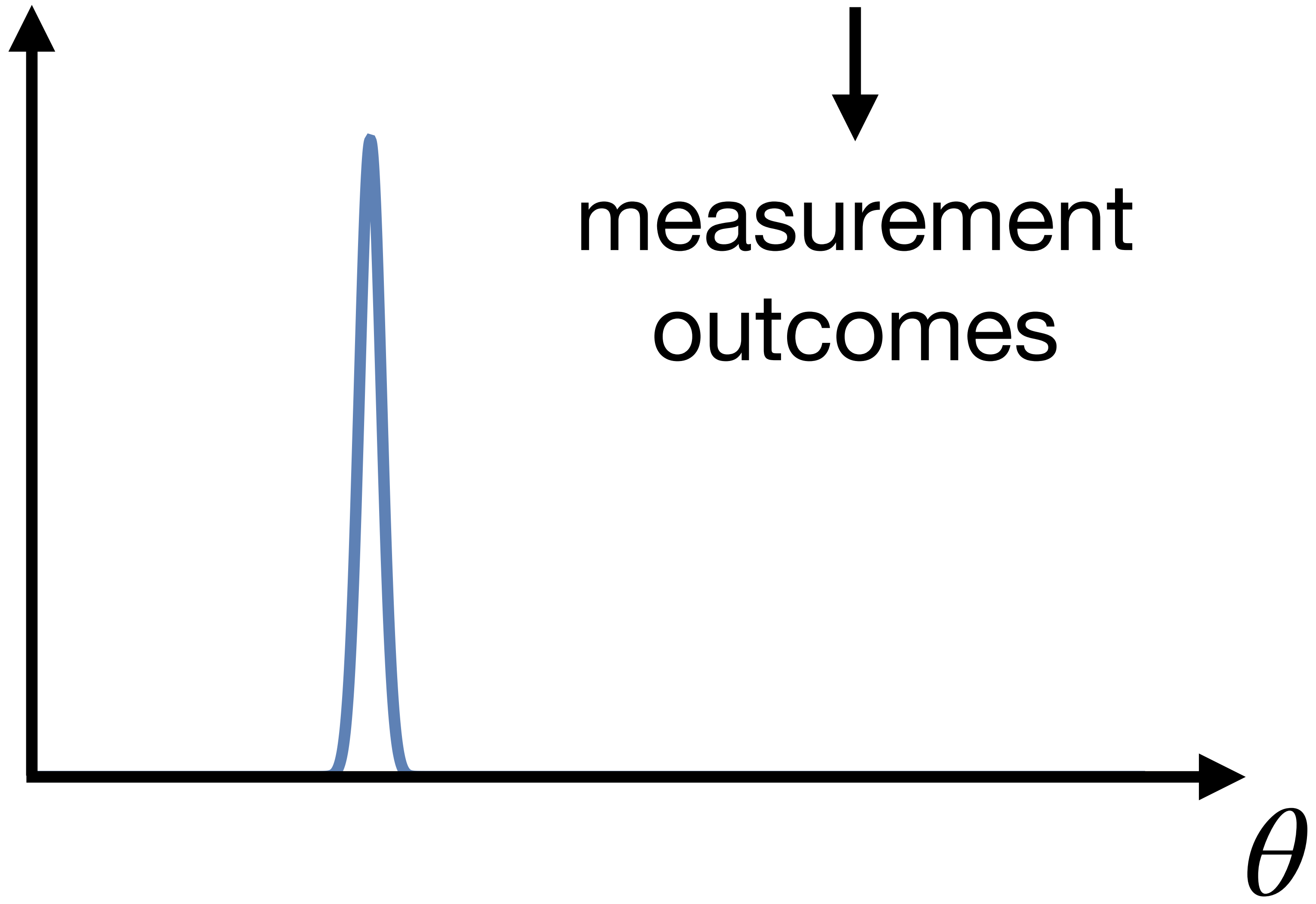


Bayesian inference →

posterior, likelihood, control parameters

$$p(\theta\,|\,\boldsymbol{x},\boldsymbol{y}) \propto p(\theta)\,p(\boldsymbol{x}\,|\,\theta,\boldsymbol{y})$$

measurement outcomes



Jaynes, *Probability Theory: The Logic of Science,* Cambridge University Press (2003)

Quantum estimation à la Bayes

Multi-shot optimisation	Single-shot optimisation
$\langle \mathscr{L}_{\boldsymbol{y}}(\boldsymbol{x}) \rangle = \int d\boldsymbol{x}\, p(\boldsymbol{x}\,\vert\,\boldsymbol{y})\, \mathscr{L}_{\boldsymbol{y}}(\boldsymbol{x})$	$\langle \mathscr{L}_{\boldsymbol{y}}(x) \rangle = \mathrm{Tr}\left\{ \int dx\, M_{\boldsymbol{y}}(x)\, W[\tilde{\theta}_{\boldsymbol{y}}(x)] \right\}$
$\mathscr{L}_{\boldsymbol{y}}(\boldsymbol{x}) = \int d\theta\, p(\theta\,\vert\,\boldsymbol{x},\boldsymbol{y})\, \mathscr{L}[\tilde{\theta}_{\boldsymbol{y}}(\boldsymbol{x}),\theta]$	$W[\tilde{\theta}_{\boldsymbol{y}}(x)] = \int d\theta\, p(\theta)\, \rho_{\boldsymbol{y}}(\theta)\, \mathscr{L}[\tilde{\theta}_{\boldsymbol{y}}(x),\theta]$
$\min_{\tilde{\theta}_{\boldsymbol{y}}(\boldsymbol{x})} \langle \mathscr{L}_{\boldsymbol{y}}(\boldsymbol{x}) \rangle$	$\min_{\tilde{\theta}_{\boldsymbol{y}}(x),\, M_{\boldsymbol{y}}(x),\, \boldsymbol{y}} \langle \mathscr{L}_{\boldsymbol{y}}(x) \rangle$
Find optimal estimator	Find optimal estimator, POM, and control parameters

Jaynes, *Probability Theory: The Logic of Science,* Cambridge University Press (2003)
Helstrom, *Quantum Detection and Estimation Theory,* Academic, New York (1976)

A new approach: symmetry-informed quantum metrology

Given $\mathscr{L}(\tilde{\theta}, \theta) = [f(\tilde{\theta}) - f(\theta)]^2$, with known f, the solutions to the aforementioned optimisation problems are as follows:

Estimator: $\tilde{\theta}_{\boldsymbol{y},f}(\boldsymbol{x}) = f^{-1}\left[\int d\theta\, p(\theta\,|\,\boldsymbol{x},\boldsymbol{y})\, f(\theta)\right]$,

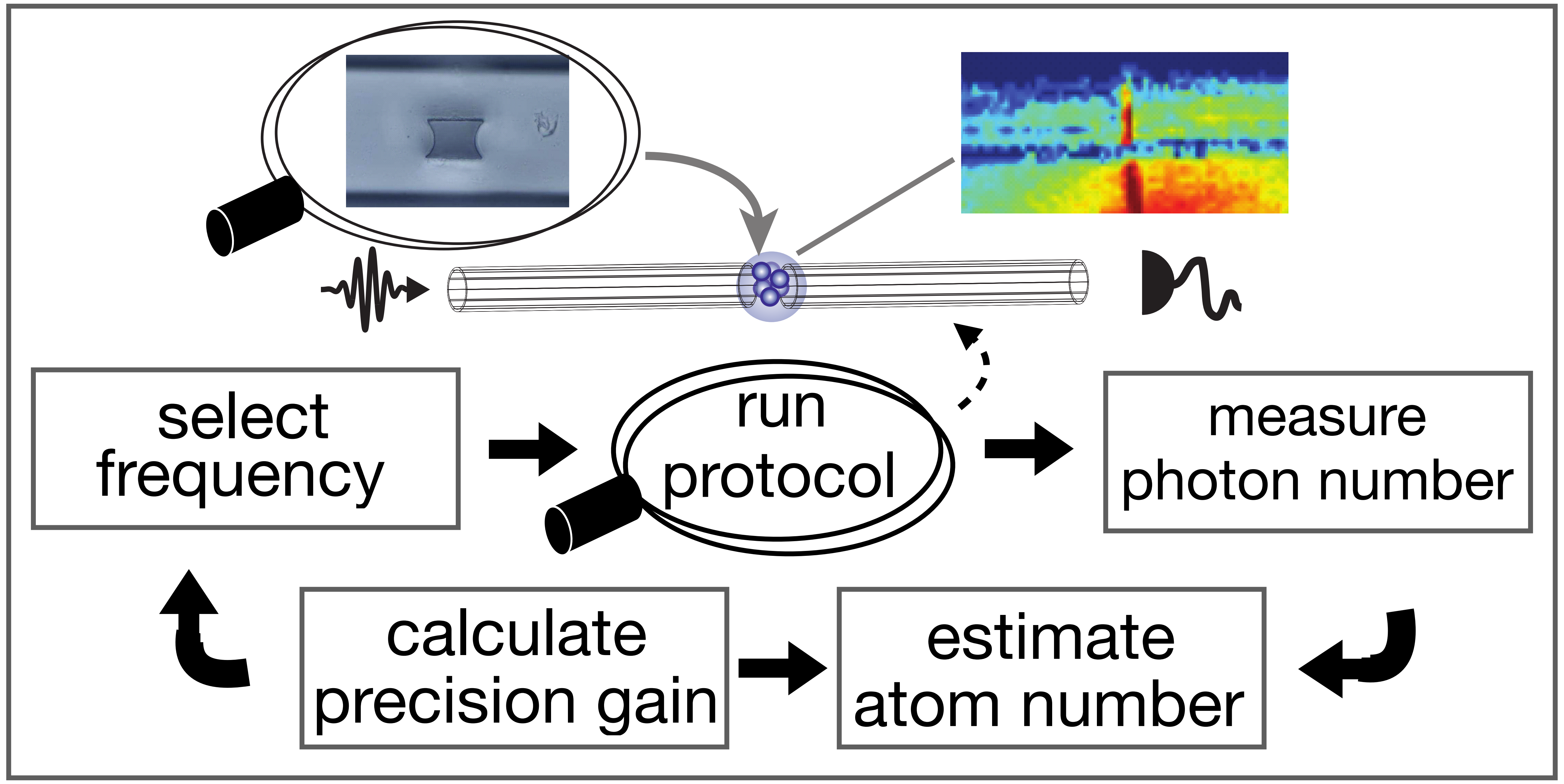
Measurement: $\mathcal{S}_{\boldsymbol{y},f}\,\rho_{\boldsymbol{y},f,0} + \rho_{\boldsymbol{y},f,0}\,\mathcal{S}_{\boldsymbol{y},f} = 2\rho_{\boldsymbol{y},f,1}$,

Precision gain: $\mathscr{G}_{\boldsymbol{y},f} = \mathrm{Tr}\left(\rho_{\boldsymbol{y},f,0}\,\mathcal{S}_{\boldsymbol{y},f}^{\;2}\right) = \int ds\, p(s\,|\,\boldsymbol{y})\, f[\tilde{\theta}_{\boldsymbol{y},f}(s)]^2$,

where $\rho_{\boldsymbol{y},f,k} = \int d\theta\, p(\theta)\, \rho_{\boldsymbol{y},f}(\theta)\, f(\theta)^k$.

Overton*, Rubio*, *et al.* arXiv:2410.10615 (2024)
Rubio, Phys. Rev. A 110, L030401 (2024)

Does it work? Evidence from cold atoms



expected number of photons with no atoms present

$$n_\nu(\varphi, \theta) = \varphi\, e^{-\zeta_\nu \theta}$$

optical depth of atomic media

laser frequency (control parameter)

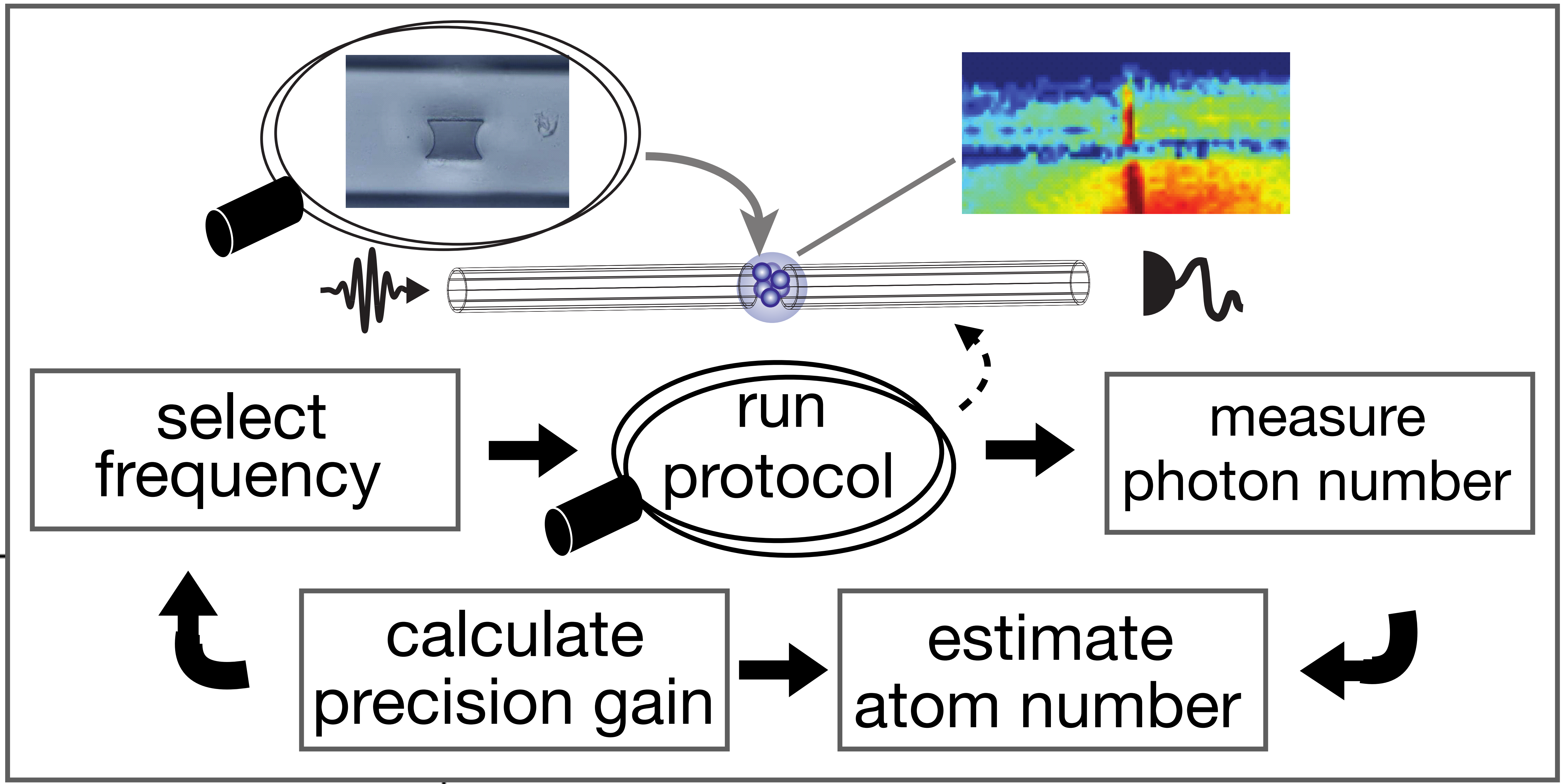
platform properties

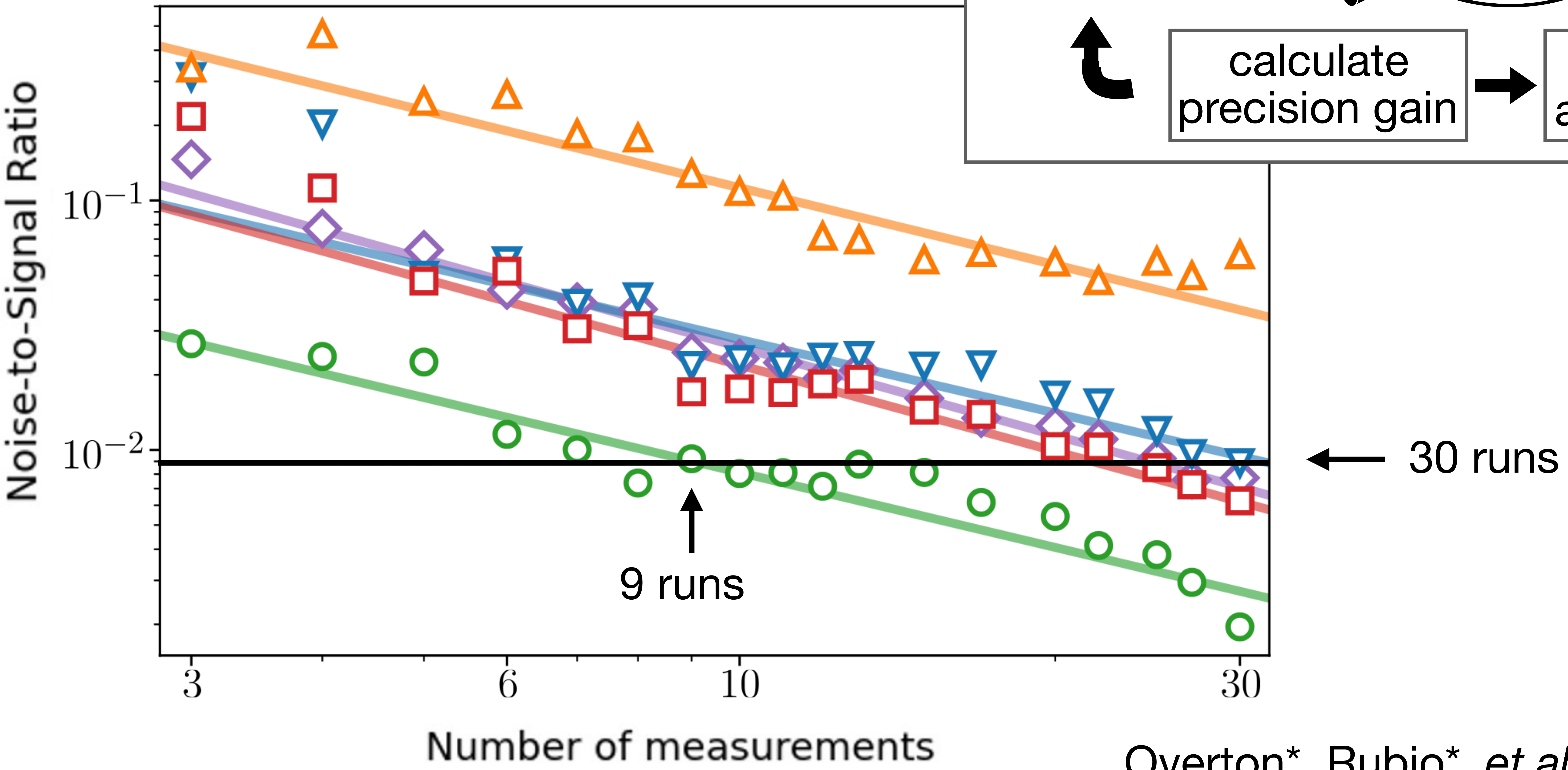
Statistical model: Poisson distribution with mean $n_\nu(\varphi, \theta)$

Target: number of atoms $N \propto \theta$

Overton*, Rubio*, *et al.* arXiv:2410.10615 (2024)

Atom number estimation





Overton*, Rubio*, *et al.* arXiv:2410.10615 (2024)

Symmetry-informed quantum metrology: under the hood

Location parameters	Location-isomorphic parameters
$\theta \in (-\infty, \infty)$	$f(\theta) \in (-\infty, \infty)$
$\theta' = \theta + \gamma$	$f(\theta') = f(\theta) + \gamma$
$p(\theta) \propto 1$	$p(\theta) \propto \frac{df(\theta)}{d\theta}$
$\mathscr{L}(\tilde{\theta}, \theta) = (\tilde{\theta} - \theta)^2$	$\mathscr{L}(\tilde{\theta}, \theta) = [f(\tilde{\theta}) - f(\theta)]^2$
$\mathcal{S}_{\mathbf{y}}\, \rho_{\mathbf{y},0} + \rho_{\mathbf{y},0}\, \mathcal{S}_{\mathbf{y}} = 2\rho_{\mathbf{y},1}$	$\mathcal{S}_{\mathbf{y},f}\, \rho_{\mathbf{y},0,f} + \rho_{\mathbf{y},0,f}\, \mathcal{S}_{\mathbf{y},f} = 2\rho_{\mathbf{y},1,f}$
$\rho_{\mathbf{y},k} = \int d\theta\, p(\theta)\, \rho_{\mathbf{y}}(\theta)\, \theta^k$	$\rho_{\mathbf{y},k,f} = \int d\theta\, p(\theta)\, \rho_{\mathbf{y},f}(\theta)\, f(\theta)^k$

Rubio, Phys. Rev. A 110, L030401 (2024)

Personick, IEEE Trans. Inf. Theory 17, 240 (1971)

Symmetry function for atom number estimation

Step 1: calculate a relevant ignorance prior

First constraint	Second constraint
$n_\nu(\varphi', \theta') = n_\nu(\varphi, \theta)$ $\downarrow$ $\begin{cases} \zeta_\nu \theta' = \zeta_\nu \theta + \log(\gamma) \\ \varphi' = \gamma \varphi \end{cases}$	The laser frequency does not inform the unknown parameters
$p(\varphi', \theta' \,\vert\, \nu)\, d\varphi' d\theta' = p(\varphi, \theta \,\vert\, \nu)\, d\varphi d\theta$	$p(\varphi, \theta \,\vert\, \nu) \mapsto p(\varphi, \theta)$

$$p(\varphi, \theta \,|\, \nu) = \mathrm{e}^{-\zeta_\nu \theta}\, h(\varphi\, \mathrm{e}^{-\zeta_\nu \theta}) \longrightarrow \boxed{p(\varphi, \theta) = \left[(\theta_{\max} - \theta_{\min}) \log\left(\frac{\varphi_{\max}}{\varphi_{\min}}\right) \varphi \right]^{-1}}$$

Overton*, Rubio*, *et al.* arXiv:2410.10615 (2024)

Symmetry function for atom number estimation

Step 2: marginalise over the nuisance parameter

$$p(\theta) = \int_{\varphi_{\min}}^{\varphi_{\max}} d\varphi \, p(\varphi, \theta) = \frac{1}{\theta_{\max} - \theta_{\min}} \propto 1$$

Step 3: equate the ignorance prior for the platform to the ignorance prior of symmetry-informed estimation

$$p(\theta) \propto \frac{df(\theta)}{d\theta} \propto 1 \longrightarrow \boxed{f(z) = c_1 z + c_2}$$

Overton*, Rubio*, *et al.* arXiv:2410.10615 (2024)

An old friend: global quantum thermometry

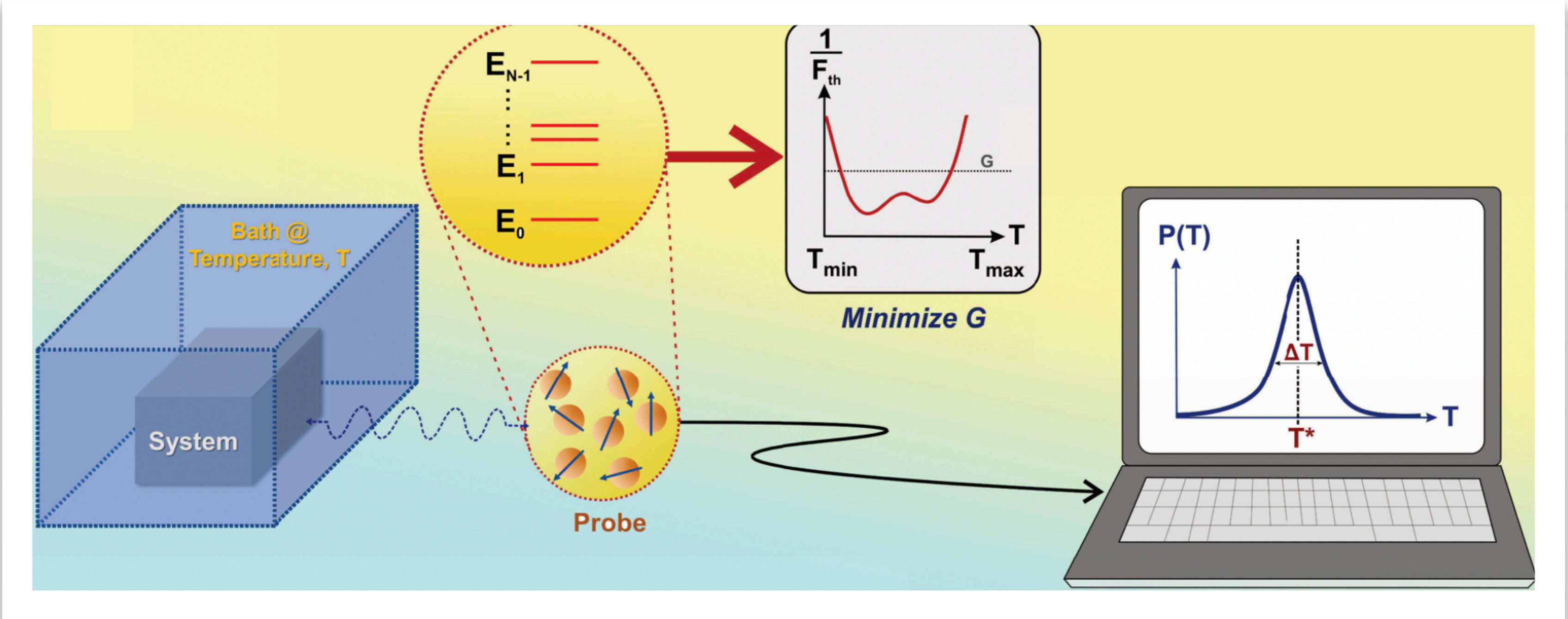
Given $f(z) = \log\left(\frac{z}{z_0}\right)$,

Estimator: $\tilde{\theta}_{\boldsymbol{y}}(\boldsymbol{x}) = \theta_u \exp\left[\int d\theta\, p(\theta|\boldsymbol{x},\boldsymbol{y}) \log\left(\frac{\theta}{\theta_u}\right)\right]$,

Measurement: $\mathcal{S}_{\boldsymbol{y}}\, \rho_{\boldsymbol{y},0} + \rho_{\boldsymbol{y},0}\, \mathcal{S}_{\boldsymbol{y}} = 2\rho_{\boldsymbol{y},1}$,

Precision gain: $\mathcal{G}_{\boldsymbol{y}} = \mathrm{Tr}\left(\rho_{\boldsymbol{y},0}\, \mathcal{S}_{\boldsymbol{y}}{}^2\right) = \int ds\, p(s|\boldsymbol{y}) \log^2\left[\frac{\tilde{\theta}_{\boldsymbol{y}}(s)}{\theta_u}\right]$,

where $\rho_{\boldsymbol{y},k} = \int d\theta\, p(\theta)\, \rho_{\boldsymbol{y}}(\theta) \log^k\left(\frac{\theta}{\theta_u}\right)$.

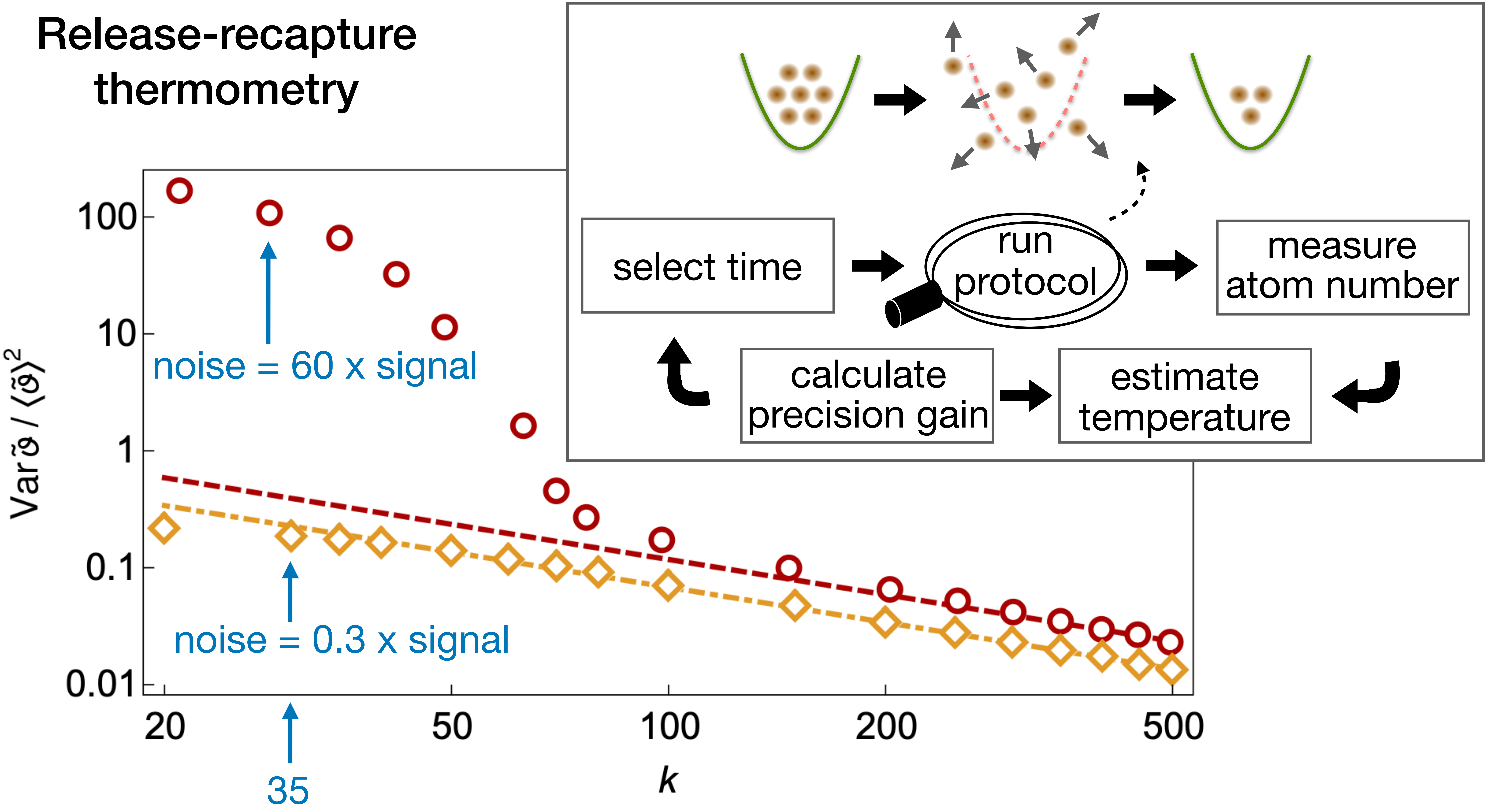


(This image: Mok et al., Commun. Phys. 4:62, 2021)

Rubio, Quantum Sci. Technol. 8, 015009 (2022)

Rubio *et al.*, Phys. Rev. Lett. 127, 190402 (2021)

Release-recapture thermometry



Glatthard, Rubio *et al.*, PRX Quantum 3, 040330 (2022)

Some implications

- **The FI is *not* needed for the optimisation of experimental sensing platforms**, and its use can be counterproductive.
- **The FI is *not* needed to search for fundamental precision limits**, although its calculation is still of interest as a measure of sensitivity.
- **Priors with logical content** (i.e., no frequentist interpretation) **are meaningful both theoretically and in experiments**.

The multiparameter challenge

- Error bounds **ignoring incompatibility:**

$$\langle \mathscr{L}_{\boldsymbol{y},\,\mathrm{lin}}(x)\rangle \geq \sum_{i=1}^{d} w_i \left[\int d\boldsymbol{\theta}\, p(\boldsymbol{\theta})\, \theta_i^2 - \mathrm{Tr}\left(\rho_{\boldsymbol{y},\,\mathrm{lin},\,0}\, \mathcal{S}^2_{\boldsymbol{y},\,\mathrm{lin}}\right)\right]$$

Rubio & Dunningham, Phys. Rev. A 101, 032114 (2020)

$$\langle \mathscr{L}_{\boldsymbol{y},\,\log}(x)\rangle \geq \sum_{i=1}^{d} w_i \left[\int d\boldsymbol{\theta}\, p(\boldsymbol{\theta})\, \log^2\left(\frac{\theta_i}{\theta_u}\right) - \mathrm{Tr}\left(\rho_{\boldsymbol{y},\,\log,\,0}\, \mathcal{S}^2_{\boldsymbol{y},\,\log}\right)\right]$$

Rubio, Quantum Sci. Technol. 8, 015009 (2022)

- Error bounds **with partial information about incompatibility:**

⟶ Suzuki, IEICE Trans. Fundamentals 107, 510 (2024)

- Exact numerical approaches:

⟶ Bavaresco *et al.*, Phys. Rev. Res. 6, 023305 (2024)

Contributions

Theory

Phys. Rev. A 110, L030401 (2024)

Quantum Sci. Technol. 8, 015009 (2022)

Phys. Rev. Lett. 127, 190402 (2021)

Phys. Rev. A 101, 032114 (2020)

Experiments

arXiv:2410.10615

J. Chem. Theory Comput. 20, 1, 385-395 (2023)

PRX Quantum 3, 040330 (2022)

Take-home message
Combining quantum and Bayesian principles leads to optimality in metrology, and this can be fully realised through the use of symmetries.