# Quantum scale metrology

## Highly-precise measurements beyond phase estimation

*Central European Workshop on Quantum Optics*
*Milan, 5th July 2023*

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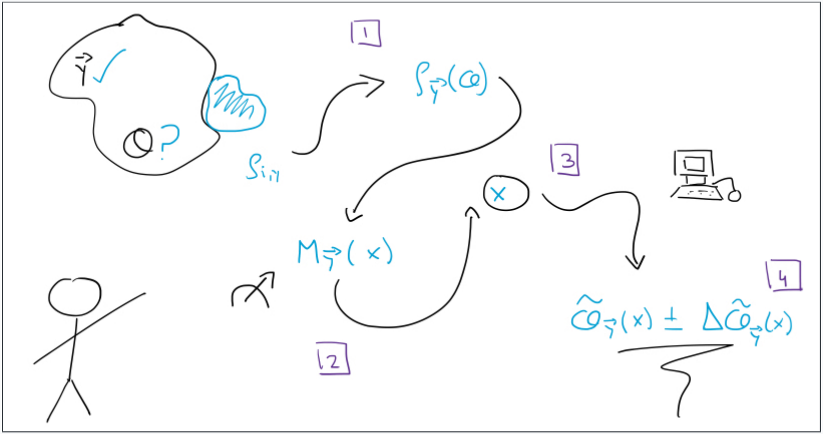
# Our plan for today

1. Quantum metrology *beyond phase estimation*
2. The implications of scale invariance
3. Optimal strategies for scale estimation
4. A case study: atomic lifetimes



# Quantum metrology *beyond phase estimation*



## Quantum metrology: fundamental problem

Minimise the *error functional*

$$\bar{\epsilon}_{\boldsymbol{y}} = \int d\theta dx\, p(\theta) \mathrm{Tr}[\rho_{\boldsymbol{y}}(\theta) M_{\boldsymbol{y}}(x)] \mathcal{D}[\tilde{\theta}_{\boldsymbol{y}}(x), \theta]$$

w.r.t. $\tilde{\theta}_{\boldsymbol{y}}(x)$, $M_{\boldsymbol{y}}(x)$ for given $p(\theta)$, $\rho_{\boldsymbol{y}}(\theta)$.

| Setup | Estimation | Probabilities |
|---|---|---|
| $x \equiv$ measurand | $\theta \equiv$ hypothesis | $p(\theta) \equiv$ prior |
| $\boldsymbol{y} \equiv$ calibration | $\tilde{\theta}_{\boldsymbol{y}}(x) \equiv$ estimator | $\rho_{\boldsymbol{y}}(\theta) \equiv$ state |
| $\Theta \equiv$ unknown | $\mathcal{D}[\tilde{\theta}_{\boldsymbol{y}}(x), \theta] \equiv$ deviation | $M_{\boldsymbol{y}}(x) \equiv$ POM |

## Quantum metrology: ultimate precision limits

Let $\tilde{\vartheta}_{\boldsymbol{y}}(x)$, $\mathcal{M}_{\boldsymbol{y}}(x)$ be the optimal strategy resulting from the minimisation problem above. Then,

$$\bar{\epsilon}_{\boldsymbol{y}} \geqslant \bar{\epsilon}_{\boldsymbol{y}}|_{\tilde{\vartheta}(x)} \geqslant \bar{\epsilon}_{\boldsymbol{y}}|_{\tilde{\vartheta}_{\boldsymbol{y}}(x),\, \mathcal{M}_{\boldsymbol{y}}(x)}.$$

## Quantum metrology: optimal data processing

$$\tilde{\vartheta}_{\boldsymbol{y}}(x) \pm \Delta\tilde{\vartheta}_{\boldsymbol{y}}(x),$$

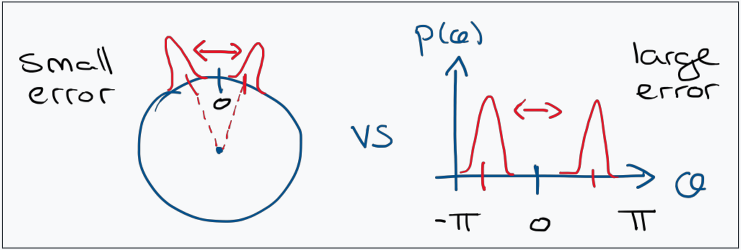
where $\Delta\tilde{\vartheta}_{\boldsymbol{y}}(x)$ is a suitable function of

$$\bar{\epsilon}_{\boldsymbol{y}}(x) = \int d\theta\, p(\theta|x, \boldsymbol{y})\mathcal{D}[\tilde{\theta}_{\boldsymbol{y}}(x), \theta],$$

with $p(\theta|x, \boldsymbol{y}) \propto p(\theta)\mathrm{Tr}[\rho_{\boldsymbol{y}}(\theta)M_{\boldsymbol{y}}(x)]$ ($\equiv$ *Bayes theorem*).

# Typical metrology frameworks

| **Parameter** | phase | location |
|---|---|---|
| **Support** | $0 \leqslant \theta < 2\pi$ | $-\infty < \theta < \infty$ |
| **Symmetry** | $\theta \mapsto \theta' = \theta + 2\gamma\pi,\ \gamma \in \mathbb{Z}$ | $\theta \mapsto \theta' = \theta + \gamma,\ \gamma \in \mathbb{R}$ |
| **Ignorance** | $p(\theta) = 1/2\pi$ | $p(\theta) \propto 1$ |
| **Error** $\mathcal{D}(\tilde{\theta}, \theta)$ | $4\sin^2[(\tilde{\theta} - \theta)/2]$ | $(\tilde{\theta} - \theta)^2$ |



Remarks:

- The formulation of quantum metrology is greatly simplified by taking the notion of *error functional* as a primitive.
- The universality of this approach is apparent inasmuch as probability theory is an extension of propositional logic.
- **Different types of parameters demand different estimation-theoretic frameworks**.
- Phase estimation is just one of many.

# The implications of scale invariance

**Definition: scale parameter**

Let $\boldsymbol{z} = (x, \boldsymbol{y})$. $\Theta \in (0, \infty)$ scales $z_i$ if $z_i$ is considered 'large' when $z_i/\Theta \gg 1$ and 'small' when $z_i/\Theta \ll 1$. This is **invariant under transformations**

$$z_i \mapsto z_i' = \gamma z_i, \;\; \Theta \mapsto \Theta' = \gamma \Theta,$$

with positive $\gamma$, since $z_i/\Theta = z_i'/\Theta'$.

**Examples:**

- Light speed: $v/c$
- Lifetime: $t/\tau$
- Rate: $kt$
- Temperature: $E/(k_B T)$ or $\beta E$

Quantum Sci. Technol. **8**, 015009 (2022)

## Maximum ignorance about scale parameters

- Alice and Bob wish to estimate $\Theta$. They are told $\Theta$ is a *scale parameter*, but they are completely ignorant otherwise.
- Alice encodes her information in $p(\theta)d\theta$, while Bob does so in $p(\theta')d\theta'$, where $\theta' = \gamma\theta$.
- Since they hold the *same* information, $p(\theta)d\theta = p(\theta')d\theta'$, i.e.,

$$p(\theta) = \gamma p(\gamma\theta).$$

The solution is:

**Jaynes's transformation groups: Jeffreys's prior**

$$p(\theta) \propto 1/\theta$$

uniquely represents maximum ignorance about scales.

IEEE Trans Cybern **4**, 227–241 (1968)

## The logarithmic error family

- Let $\phi \in (-\infty, \infty)$ be a location parameter.
- By virtue of translation invariance, $\mathcal{D}(\tilde{\phi}, \phi) = |\tilde{\phi} - \phi|^k$ and maximum ignorance is represented by $p(\phi) \propto 1$.
- This scenario can be mapped to scale estimation by setting $\phi = \alpha \log(\theta/\theta_u)$, where $\alpha$, $\theta_u$ are free parameters.
- That is, $p(\phi)d\phi = p(\theta)d\theta$ implies $p(\phi) \propto 1 \mapsto p(\theta) \propto 1/\theta$.

Therefore:

**Deviation function: the logarithmic family**

$$\mathcal{D}(\tilde{\theta}, \theta) = |\alpha \log(\tilde{\theta}/\theta)|^k$$

Phys. Rev. Lett. **127**, 190402 (2021)

## Deviation function: properties of the logarithmic family

- Scale invariant, i.e., $\mathcal{D}(\gamma\tilde{\theta}, \gamma\theta) = \mathcal{D}(\tilde{\theta}, \theta)$.
- Symmetric, i.e., $\mathcal{D}(\tilde{\theta}, \theta) = \mathcal{D}(\theta, \tilde{\theta})$.
- Reaches its absolute minimum at $\tilde{\theta} = \theta$, where it vanishes.
- Grows (decreases) monotonically from (towards) that minimum when $\tilde{\theta} > \theta$ ($\tilde{\theta} < \theta$).
- Can be interpreted as a generalised **noise-to-signal ratio** when $\alpha = 1$, $k = 2$, i.e.,

$$\mathcal{D}(\tilde{\theta}, \theta) = \log^2(\tilde{\theta}/\theta).$$

Phys. Rev. Lett. **127**, 190402 (2021)

## Why does the logarithmic error family matter?

- Alice and Bob wish to estimate $\Theta$, with $\theta \in [0.01, 100]$.
- Alice is not sure about scale estimation, so she continues to use $p(\theta) \propto 1$ and minimises $\int d\theta\, p(\theta)(\tilde{\theta} - \theta)^2$, finding

$$\tilde{\theta} = \int d\theta p(\theta)\theta \simeq 50.$$

- Bob, on the other hand, uses $p(\theta) \propto 1/\theta$ and minimises $\int d\theta\, p(\theta) \log^2(\tilde{\theta}/\theta)$, finding

$$\tilde{\theta} = \theta_u \exp\left[\int d\theta p(\theta) \log\left(\frac{\theta}{\theta_u}\right)\right] = 1.$$

- $\tilde{\theta} = 1$ is the *middle point w.r.t. the orders of magnitude within the prior range*, and so the correct answer.

In summary:

| | |
|---|---|
| **Parameter** | scale |
| **Support** | $0 < \theta < \infty$ |
| **Symmetry** | $\theta \mapsto \theta' = \gamma\theta,\ \gamma \in \mathbb{R}_{++}$ |
| **Ignorance** | $p(\theta) \propto 1/\theta$ |
| **Error** $\mathcal{D}(\tilde{\theta}, \theta)$ | $\log^2(\tilde{\theta}/\theta)$ |

# Optimal strategies for scale estimation

**Quantum scale metrology: fundamental problem**

Minimise the *mean logarithmic error*

$$\bar{\epsilon}_{\boldsymbol{y},\mathrm{mle}} = \int d\theta dx\, p(\theta)\mathrm{Tr}[\rho_{\boldsymbol{y}}(\theta)M_{\boldsymbol{y}}(x)]\,\log^2\left[\frac{\tilde{\theta}_{\boldsymbol{y}}(x)}{\theta}\right]$$

w.r.t. $\tilde{\theta}_{\boldsymbol{y}}(x)$, $M_{\boldsymbol{y}}(x)$ for given $p(\theta)$, $\rho_{\boldsymbol{y}}(\theta)$. Here, it is assumed that: (i) $p(\theta|\boldsymbol{y}) \mapsto p(\theta)$, and (ii) $\Theta$ scales $\boldsymbol{y}$, but not $x$, which is dimensionless.

This can be solved *analytically* via **Jensen's operator inequality and** an operator version of **the calculus of variations**.

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## Result 1: optimal strategy

Let $\mathcal{S}_{\boldsymbol{y}} = \int ds\, \mathcal{P}_{\boldsymbol{y}}(s)\, s$ solve the *Lyaponuv equation*

$$\mathcal{S}_{\boldsymbol{y}} \rho_{\boldsymbol{y},0} + \rho_{\boldsymbol{y},0} \mathcal{S}_{\boldsymbol{y}} = 2\rho_{\boldsymbol{y},1},$$

where

$$\rho_{\boldsymbol{y},k} = \int d\theta\, p(\theta) \rho_{\boldsymbol{y}}(\theta) \log^k \left( \frac{\theta}{\theta_u} \right);$$

then, the **optimal estimator** is

$$\tilde{\theta}_{\boldsymbol{y}}(x) \mapsto \tilde{\vartheta}_{\boldsymbol{y}}(s) = \theta_u \exp(s),$$

and the **optimal POM** is

$$M_{\boldsymbol{y}}(x) \mapsto \mathcal{M}_{\boldsymbol{y}}(s) = \mathcal{P}_{\boldsymbol{y}}(s).$$

## Result 2: ultimate precision limits

$$\bar{\epsilon}_{\boldsymbol{y},\mathrm{mle}} \geqslant \bar{\epsilon}_p - \mathcal{K}_{\boldsymbol{y}} \geqslant \bar{\epsilon}_p - \mathcal{J}_{\boldsymbol{y}}$$

| | | |
|---|---|---|
| $\bar{\epsilon}_p$ | $\int d\theta\, p(\theta) \log^2(\theta/\tilde{\vartheta}_p)$ | prior error |
| $\tilde{\vartheta}_p$ | $\theta_u \exp[\int d\theta\, p(\theta) \log(\theta/\theta_u)]$ | prior estimate |
| $\mathcal{K}_{\boldsymbol{y}}$ | $\int dx \{\mathrm{Tr}[M_{\boldsymbol{y}}(x)\rho_{\boldsymbol{y},1}]^2/\mathrm{Tr}[M_{\boldsymbol{y}}(x)\rho_{\boldsymbol{y},0}]\}$ | classical IG* |
| $\mathcal{J}_{\boldsymbol{y}}$ | $\mathrm{Tr}(\rho_{\boldsymbol{y},0}\mathcal{S}_{\boldsymbol{y}}^2) = \mathrm{Tr}(\rho_{\boldsymbol{y},1}\mathcal{S}_{\boldsymbol{y}})$ | quantum IG* |

*IG $\equiv$ information gain

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## Result 3: optimal data processing

$$\tilde{\vartheta}_{\boldsymbol{y}}(s) \pm \Delta\tilde{\vartheta}_{\boldsymbol{y}}(s) = \tilde{\vartheta}(s)[1 \pm \bar{\epsilon}^{1/2}_{\boldsymbol{y},\mathrm{mle}}(s)],$$

where

$$\tilde{\vartheta}_{\boldsymbol{y}}(s) = \theta_u \exp\left[\int d\theta\, p(\theta|s,\boldsymbol{y}) \log\left(\frac{\theta}{\theta_u}\right)\right]$$

and

$$\bar{\epsilon}_{\boldsymbol{y},\mathrm{mle}}(s) = \int d\theta\, p(\theta|s,\boldsymbol{y}) \log^2\left(\frac{\theta}{\theta_u}\right) - \tilde{\vartheta}^2_{\boldsymbol{y}}(s),$$

with $p(\theta|s,\boldsymbol{y}) \propto p(\theta)\mathrm{Tr}[\rho_{\boldsymbol{y}}(\theta)\mathcal{P}_{\boldsymbol{y}}(s)]$ ($\equiv$ *Bayes theorem*).

Quantum Sci. Technol. **8**, 015009 (2022)
PRX Quantum **3**, 040330 (2022)

Remarks:

- **We can now calculate *universally optimal* estimators and POMs** for any given prior and state in scale metrology.
- **This enables the search for ultimate precision limits and optimal protocols** for data analysis in scale metrology.

# A case study: atomic lifetimes

- Let a two-level atom prepared as $|\psi\rangle = \sqrt{1-a}\,|g\rangle + \sqrt{a}\,|e\rangle$ undergo *spontaneous photon emission*.
- Such a process may be described as

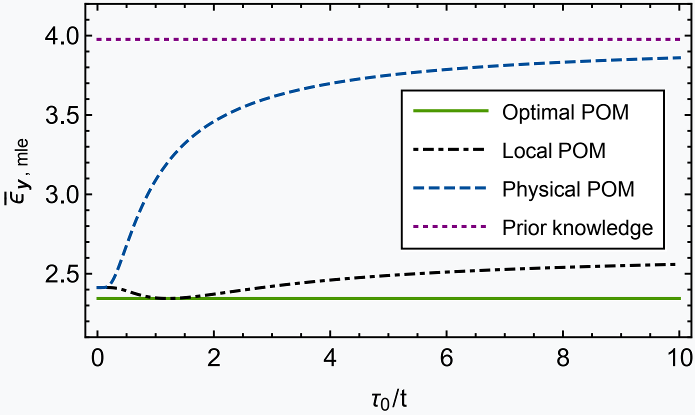
$$\rho_t(\tau) = \begin{pmatrix} [1 - a\,\eta_t(\tau)] & [a(1-a)\,\eta_t(\tau)]^{\frac{1}{2}} \\ [a(1-a)\,\eta_t(\tau)]^{\frac{1}{2}} & a\,\eta_t(\tau) \end{pmatrix},$$

with $\eta_t(\tau) = \exp(-t/\tau)$, **lifetime** $\tau$ and elapsed time $t$.

**Problem: quantum estimation of a time scale**

Unknown parameter: $\Theta = \tau$; prior information: $\theta/t \in [0.01, 10]$, $a = 0.9$.

S. M. Barnett, *Quantum Information*, Oxford: Oxford University Press (2009)




- **'Yes'/'No' measurement**: $M^{Y}_{t,\tau_\circ} = [1 - \eta_t(\tau_\circ)]\, |e\rangle\langle e|$ ('Yes'), $M^{N}_{t,\tau_\circ} = |g\rangle\langle g| + \eta_t(\tau_\circ)\, |e\rangle\langle e|$ ('No').
- Informative (reduces $\bar{\epsilon}_p$), but $\tau$ easier to estimate when decay likely to have already happened ($\tau_\circ/t \ll 1$).
- Initial 'hint' $\tau_o$ needed, and generally suboptimal.

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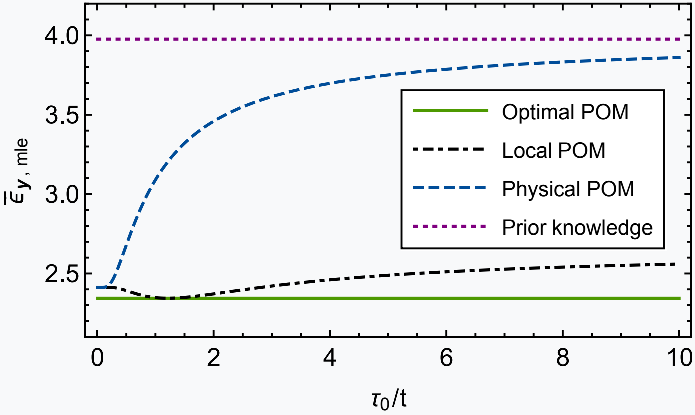


- **SLD measurement**: $M^i_{t,\tau_\circ} = |\lambda^i_{t,\tau_\circ}\rangle\langle\lambda^i_{t,\tau_\circ}|$, with $L_t(\tau_\circ)|\lambda^i_{t,\tau_\circ}\rangle = \lambda^i_{t,\tau_\circ}|\lambda^i_{t,\tau_\circ}\rangle$ and $L_t(\tau)\rho_t(\tau) + \rho_t(\tau)L_t(\tau) = 2\partial_\tau\rho_t(\tau)$.
- More informative than 'Yes'/'No' measurement.
- Initial 'hint' $\tau_o$ still needed, and suboptimal for $\tau_\circ/t \gg 1$.

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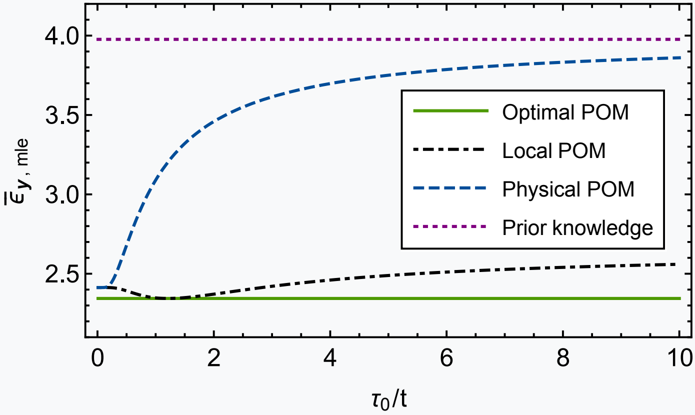


- **Optimal measurement**: $|\psi_+\rangle = 0.094\,|g\rangle + 0.996\,|e\rangle$, $|\psi_-\rangle = 0.996\,|g\rangle - 0.094\,|e\rangle$.
- Globally optimal ($\tau_0$-independent).
- Establishes the fundamental precision limit for the estimation of $\tau$.

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Remarks:

- Scale metrology enables the possibility of **exploiting quantum resources to estimate time and other scales**.
- Moreover, it can establish fundamental precision limits **in the presence of finite prior information**.
- It may be argued that local estimation theory, while valid and useful in its regime of applicability, is not essential.

# Epilogue: multiparameter estimation à la Bayes

**Multiparameter metrology: scales**

$$\bar{\epsilon}_{\boldsymbol{y},\mathrm{mle}} \geqslant \sum_i w_i \left[ \int d\Theta\, p(\Theta) \log^2 \left( \frac{\theta_i}{\theta_{u,i}} \right) - \mathrm{Tr}(\rho^{\mathrm{mle}}_{\boldsymbol{y},1,i} \mathcal{S}^{\mathrm{mle}}_{\boldsymbol{y},i}) \right]$$

**Multiparameter metrology: locations**

$$\bar{\epsilon}_{\boldsymbol{y},\mathrm{mse}} \geqslant \sum_i w_i \left[ \int d\Theta\, p(\Theta) \theta_i^2 - \mathrm{Tr}(\rho^{\mathrm{mse}}_{\boldsymbol{y},1,i} \mathcal{S}^{\mathrm{mse}}_{\boldsymbol{y},i}) \right]$$

- $w_i \equiv$ importance weight for the $i$th parameter
- Not saturable when $[\mathcal{S}_{\boldsymbol{y},i}, \mathcal{S}_{\boldsymbol{y},j}] \neq 0$ for $i \neq j$
- Starting point to study an **uncertainty- and prior-dependent notion of quantum incompatibility**

Quantum Sci. Technol. **8**, 015009 (2022)
Phys. Rev. A **101**, 032114 (2020)

## Conclusions:

- Scale metrology **enables the most precise estimation of scale parameters that is allowed by quantum mechanics**.
- It provides a more fundamental picture of metrology, while also being practical and easy to use.
- It **opens the door to constructing new quantum estimation theories for all kinds of parameters**.

**Key works:**


Quantum Sci. Technol. **8**, 015009 (2022)
Phys. Rev. Lett. **127**, 190402 (2021)
Phys. Rev. A **101**, 032114 (2020)